First impressions on Suntola's expanding $S^3(r_0)$ cosmological model: A motivational study surveying the Dynamic Universe*

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Starting from the constraints of an expanding $S^3(r_0)$ geometry (3-sphere), which is locally Euclidean but globally closed (spatially circular), Suntola has arrived at a cosmological model where the expansion of the universe and the existence of a singularity in the past is only natural. Due to the chosen geometry and a few additional constraints, several metrics that are commonly used in cosmology, such as comoving distance, light travel distance, angular diameter distance, and luminosity distance, get particularly simple and elegant mathematical forms. Besides conservation of energy and momentum, conservation of abstract mass has been the most important principle guiding the work. To make the model formally coherent and minimal in that vein, however, the model suggests the speed of light to actually vary both over astronomical and local distances, and furthermore, the model does not assume the invariance of rest mass to apply generally. Surprisingly, these concessions seem to make the model consistent with a wide range of observations, including relativistic phenomena, without dilated time or contracted distance. The model addresses the most common concerns directed at similar extremely speculative studies by keeping the mathematics particularly straightforward but still capable of generalization, and by aiming to face the existing cosmological records and published experiments as in depth as practically possible for this kind of a novel theoretical endeavor so far.

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I. AN EXPANDING S^3 GEOMETRY AS A BASELINE: PERSONAL OBSERVATIONS

I have been studying DU—a rather grandiosely named dynamic universe model proposed by Suntola (2018, 2021)—for a few weeks now since the event here in Helsinki[1], and been fairly impressed by the simple elegance of the formulas derived. It is also very fortunate to be able to discuss with Suntola directly about the specifics.

Being familiar with some university level maths and physics, my first target was to understand the overall

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[1] The Finnish Society for Natural Philosophy organized an event at the House of Science and Letters, Helsinki, on the 24th of January 2023, where Suntola presented his work. The theme of the event was "Is critiquing relativity theory science denial?", and all the participants seemed to be in agreement that theoretical and experimental work is always welcome to address potential shortcomings of any established theory.

structure and some details of this proposed cosmological model, to be able to later prepare and submit some theoretical communications for review, in some appropriate context. So far, there are reasons to believe that this could prove out to be a novel way of looking at things, potentially even unifying in the long run, so it seems certainly worth looking into.

The way I see it, Suntola has started developing his model directly from the constraints of the geometry of $S^3(r_0)$ (three-sphere), which was conjectured already by Poincaré to be in some sense the simplest 3-space there is. S^3 is a compact, connected, 3-dimensional manifold without boundary, and also simply connected ("yhdesti yhtenäinen" in Finnish), so it seems like a very sensible starting point for modeling the observed universe, provided the actual physics are kept in mind (as far as humanly possible for one person at this point).

The conjecture that the 3-sphere is the only three-dimensional manifold (up to topological homeomorphism) with the aforementioned properties, was proved only about 20 years ago by Grigori Perelman, developing some pretty hefty mathematical tools in the process to be able to make progress (building on Ricci flow developed by Richard Hamilton in the 1980s, among other results). Of course, that by itself does not prove much for physics, but lends some credibility to the approach studied here.

For a nice story about the Poincaré conjecture, see "The shapes of space" (Collins, 2004), which ends with the phrasing: "It remains to be seen if [Perelman's] techniques will reveal interesting new information about general relativity or string theory. If that is the case, Perelman will have taught us not only about the shapes of abstract 3-spaces but also about the shape of the particular space in which we live." I do not mean to particularly highlight the tools employed such as the Ricci flow (which I am not at all aiming to use here, even though somebody may be able to apply the formal apparatus of renormalization group at different scales here), but the fundamentals of the S^3 geometry.

In physics, S^3 (as a unit sphere version, as opposed to Suntola's expanding one) is mostly known in particle physics as a Lie group (as unit quaternions) isomorphic to SU(2), which is a crucial symmetry group in the electroweak interactions. The breakage of SU(2) symmetry also results in Higgs bosons, related to how particles are assumed to gain mass. There are all kinds of known connections to 3D-rotations in ordinary 3-space via quaternions, for example, but as a cosmological starting point, S^3 is perhaps still underscrutinized [see also Peterson (1979)]). A cursory literature search hints that it has been in the minds of many people of the past—as Suntola acknowledges, too—with different terms and meanings.

Of course, as a very fundamental geometrical object, the specific S^3 surely is discussed in the cosmological literature, but from a quick look, in a very different setting than what Suntola argues is the most decisive one—Suntola presents $S^3(r_0)$ as a somewhat malleable expanding object in a rigidly Euclidean four-space, not spacetime. Call it a hyperspace if you will.

I looked briefly into Sean Carroll's lecture notes on general relativity (Carroll, 1997), and there the three-sphere is mentioned on pages 202 and 226, around which there could be something relevant to these discussions too. There are similar mathematical forms, but they may be mostly coincidences due to Suntola arriving at very similar forms as Schwarzchild when modeling the free fall in the gravitational potential from a conservation of energy point of view, while not using (and even actively avoiding) the spacetime concept. There are also products of cosine terms on page 202 and elsewhere, which surely are coincidences too, but they attract my attention somewhat due to DU having cosine factors as central elements in the model—more about this later.

There is also an unverified quote floating in the internet attributed to S. Carroll stating that "the only possible global structure [of the cosmos] is the complete three-sphere", but "he does not go into why." If all the locations in space are at the same "distance" from the singularity at Big Bang, then a spherical geometry such as $S^3(r_0)$ seems like a very sound idea.

Instead of starting from a field theory such as Einstein's [see Janssen *et al.* (2007) for a historical account on the ups and downs while formulating general relativity], it seems that Suntola has been aiming to match the observed reality by adding just a few specific ideas to accompany this expanding $S^3(r_0)$ geometry, to give it the most relevant physical properties as he has deemed best. Many of the proposed ideas are certainly very bold and unconventional indeed, but looking at the resulting model they start making sense. The ideas are mathematically rather reasonable in their simplicity, and as such may prove out to be even obvious in retrospect if they turn out to be right.

As I expressed my first impressions during the first few weeks with his work: "Mathematical elegance has often guided finding new physics in the past."

II. THE MATHEMATICS OF THE MODEL

Considering the DU book (Suntola, 2018), let's acknowledge that Suntola approaches model development from a conservation of energy and momentum point of view, which is a very nice starting point facilitating using mostly scalars and quite straightforward vector decompositions and sums. He uses at most Gibbs-Heaviside 3D vector analysis, and manages to apply the mathematics quite effectively to a wide range of problems. I have certainly never seen celestial mechanics done in 4D as in the DU book, still matching various recorded observations very precisely.

However, from a scientific communications perspective, it is rather unfortunate that the book seems quite unaware of some of the very useful tools employed quite commonly in advanced engineering and mathematics, such as higher dimensional noncommutative algebras, and it introduces complex numbers where they are not really needed, as the "imaginary" direction. When studying the work, I feel that one should just skip the complex number notation as incidental and think in terms of 4-dimensional vectors (in some signature, perhaps simply positive definite for now). The imaginary unit is used as $i^2 = -1$ in just a few places in the book, such as when modeling Coulomb/magnetic attraction (p. 173, Box 5.1.2-A), but I have not studied those parts at length yet. Most commonly the imaginary unit is used as a label to denote the radial expansion direction (not always the actual expansion along the ray from the origin, but a local projection of it) orthogonal to the local, possibly tilted (due to the gravitational potential) Euclidean 3-space. Even better if one could start translating the equations to some suitable associative algebra, even if it is just $\mathbb{G}_3$ or $\mathbb{G}_4$, or their mixture $\mathbb{G}_{1,3}$, or perhaps $\mathbb{G}_{4,1}$. Personally I am looking forward to see some progress here in the long term, so I gathered some notes on the subject of modeling these kind of higher dimensional spaces in the appendix.

The best for now is to picture just an enormous expanding $S^3(r_0)$ hypersphere in $\mathbb{R}^4$,

$$S^3(r_0) := \{x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = r_0^2\}, \tag{1}$$

where any ray from the origin will intercept the spherical shell once at the distance r_0, and the ordinary Euclidean 3-space is orthogonal to that radial ray, there, at each point x.

The spherical 3-space does not have a boundary—if one started traveling in any direction in the three-dimensional tangential space of $S^3(r_0)$, the same place would be reached a long time later, provided one could travel fast enough compared to the expansion.[2] But now the arrival point would be much further in the hyperspace, as the sphere has expanded continuously in the meantime, the traveler gotten a ride within. Also, very interestingly, if something emits light to every direction in the 3-space, and light eventually travels all around the cosmos, the emitter could see its own image from the past, in every direction (hypothetically even from multiple round trips), reminiscent of cosmic background radiation.

Later the perfect symmetry in Eq. (1) will be relaxed by allowing the sphere to have small dents, modeling important dynamics between the motion and gravitation.

The scalar mathematics are greatly facilitated by Suntola having succeeded very impressively (in my view) in using the symmetries of S^3 in his favor (and of course the evenness of energies and thus scalars involved helps). In hindsight, of course one can draw a great circle $S^1(r_0)$,

$$S^1(r_0) := \{y \in \mathbb{R}^2 \mid y_1^2 + y_2^2 = r_0^2\}, \tag{7}$$

between any two spatial locations in $S^3(r_0)$ using the origin, and it is a very visual way to look at the cosmology thus presented, in the grand scheme of things. I presume the approach appears on a first look to many as naive, but actually on a closer examination it is simply true that one can take any two points x' and x'' in the ordinary 3-space (inside the S^3 manifold), and draw the great circle, and it will be the correct one (as all points x are at the same distance r_0 from the origin by definition, and points have unambiguous shortest standard distance in S^3, presented as the arc length).

Now, the $S^3(r_0)$ is just the global view, and to make the model work with actual data (considering the astronomical dimensional differences in scale in the range of phenomena studied in models of gravity), Suntola has had to develop his concept of nested energy frames[3], descending from the global $S^3(r_0)$ towards the mass centers under scrutiny in a multiscale fashion. But in defence of DU, this kind of discretization is needed anyway (and observed too, due to the attractive nature of gravity, resulting in quite localized concentrations of masses in space),

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[2] Traveling around the universe turns out to be theoretically possible in this model, as the velocity of expansion and the maximum tangential traveling speed c_0 are connected in a special way,

$$\frac{dr_0}{dt} = c_0 = \sqrt{\frac{GM_0}{r_0}}, \tag{2}$$

as is postulated later in the foundational Eq. (23). Specifically, starting at time t_0 (measured as time from singularity) and traveling at the maximum speed c_0 around the expanding $S^3(r_0)$ cosmos, the arrival time, using Eqs. (28) and (33), is

$$t_2 = e^{3\pi} t_0 \approx 12391.64\, t_0, \tag{3}$$

and the cosmos has expanded in the meantime to the radius

$$r_0(t_2) = e^{2\pi} r_0(t_0) \approx 535.49\, r_0(t_0). \tag{4}$$

The hypothetical light-like traveler could have celebrated the progress at the antipodal point at the opposite side of the universe already at the time instant

$$t_2 = e^{3\pi/2} t_0 \approx 111.32\, t_0, \tag{5}$$

at the expanded radius

$$r_0(t_1) = e^{\pi} r_0(t_0) \approx 23.14\, r_0(t_0). \tag{6}$$

Of course, these are some pretty big numbers, considering the already large dimensions and timescales involved. For the traveler itself, one could take into account the slowing down of the traveler's clock at high velocities—which would come to a standstill when somehow traveling at the maximum speed, similar to light as electromagnetic radiation.

[3] The terminology here has been in flux, as Suntola uses the term *energy frame* consisting of *motion* and *gravitation states*, and I have been experimenting with a bit more specific *mass and motion frame*, as gravitation is both generated and responded to by mass, and the motion and gravitation are in balance, in this model. But I prefer the term energy frame here, to keep the terminology consistent. The mathematics are the same, of course.

and in any case the radial displacements from perfect $S^3(r_0)$ seem miniscule compared to the cosmos at large. Due to this quite reasonable emphasis on the centers of mass, it remains to be seen, however, how the gravitation of fluids, gases, and dust is modeled in this framework. Suntola assures that it should be quite straightforward using mass densities, but understandably we have not yet seen any simulations here.

In the local view, I guess some of the odd spatial terms are perhaps missing (or their meaning is found later, such as some torsion in the celestial mechanics analysis, or the simplifying affordances of bivectors, trivectors, and quadrivectors, or even higher order constructs to be conceivable when thinking about the 4D-rotations involved), but the simple scalar and vector decompositions make so much visual sense, compared to some more abstract (but much needed here too) tensorial approaches. And the quite simple forms of formulas offer now good opportunities for mathematical generalization.

The clarity of concepts is of great importance. I am not claiming the model is yet clear throughout, just that the model has advanced a lot already during the years.

III. THE PROPOSED STRUCTURE OF REST ENERGY

Now the actual propositions for the model to be able to match the observations, when expressed in words, are quite intimidating in their potential implications throughout the accumulated body of knowledge in physics and cosmology.

See, e.g., Ellis (2007) for the various requirements on consistent varying speed of light theories, that of course Suntola's expanding $S^3(r_0)$ cosmological model needs to satisfy too, to be able to develop further with confidence. According to my current knowledge, DU is compliant with those requirements, but in the process has been compelled to propose some very drastic changes to the way we usually think about physics.

A. Mass-energy equivalence and expansion momentum

Suntola proposes that one of the most iconic equations in the sciences, the mass-energy equivalence $E = mc^2$, has an inner structure, related to the expansion of $S^3(r_0)$ geometry. Specifically, if we write the ordinary rest energy of mass m_0 as

$$E_0 := c_0 m_0 c \qquad [\mathrm{J} := \mathrm{kg\,m^2/s^2}], \tag{8}$$

where $c_0 := dr_0/dt = \dot{r}_0$, the current radial expansion velocity of the $S^3(r_0)$ geometry, and $c \leq c_0$ is the local speed of light and radiation in the current energy frame, we get a bilinear equation where both the rest mass m_0 and the speed of light c could formally vary to conserve the energy and momentum.

Note that at first, these are all scalars here, but each factor, especially the mass, may of course have a more fundamental geometric structure, possibly also consisting of wave vectors or other frequency-related constructs (more about this later).[4] In what follows, the subscript zero is often used as a label to denote the mathematical constructs related to the radial expansion of the $S^3(r_0)$ geometry, as opposed to constructs in the three-dimensional "ordinary space", tangential to it. It is also quite intentional to use the zero for the rest mass m_0 here.

Thus, instead of thinking that matter and radiation moves through spacetime, where c is the conversion factor, Suntola proposes that the rest energy of matter is actually related to the radial expansion movement of the $S^3(r_0)$ geometry in the 4D-hyperspace, at the speed of light.[5] It is not an unreasonable assumption, considering the chosen geometry. But as the expansion velocity c_0 is very close to the measured speed of light $c \approx 2.998 \times 10^8$ m/s here on Earth, it is quite dizzying to think that under this model, everything is racing at approximately 300 kilometers per millisecond, all the time, into this proposed hyperspace, and this has been

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[4] Also David Bohm has pondered on "the problem of rest energy" as some kind of movement. From the preface of his book on special relativity (Bohm, 1996 [1965], p. 8):

> [...] Einstein's relativistic formulas, expressing the mass and momentum of a body in terms of its velocity. By means of an analysis of these formulas, one comes to Einstein's famous relationship, $E = mc^2$, between the energy of a body and its mass. The meaning of this relationship is developed in considerable detail [in this book], with special attention being given to the problem of "rest energy," and its explanation in terms of to-and-fro movements in the internal structure of the body, taking place at lower levels.

[5] If one really needs to treat time as a separate dimension for modeling purposes, one could perhaps use the 5-dimensional $\mathbb{G}_{4,1}$ to model this expansion, as the complexified $\mathbb{G}_4$ is isomorphic to real $\mathbb{G}_{4,1}$, which is also isomorphic to $\mathbb{G}_{2,3}$ and $\mathbb{G}_{0,5}$ (which are isomorphic to complexifications of $\mathbb{G}_{2,2}$ and $\mathbb{G}_{0,4}$, respectively). See, for example, Lounesto (1986, 2001), and also Sobczyk (2019).

The first subindex p in $\mathbb{G}_{p,q}$ is the number of orthogonal basis vectors that square to one, and the second subindex q is the number of basis vectors that square to -1 ("spacelike" and "timelike" directions, depending on convention which is which). Complexification simply means that the algebra uses complex numbers instead of real numbers as a base field, and it is in many cases convenient, as four-dimensional algebra representations as real 4×4 matrices can still have imaginary eigenvalues, so using complex numbers is theoretically advantageous.

This possibility of embedding the spacetime in a five-dimensional framework is reminescent of Kaluza theories, where Klein hypothesized a compact, microscopic fifth dimension, resulting in Kaluza-Klein theories [for one historical account, see the introduction in Williams (2020), also see perhaps Gogberashvili (2002)]. But now in Suntola's model, the "fifth dimension" is actually primary, and it is not microscopic, but about as macroscopic as one can be.

going on for quite some time—and that it turns out that the model predicts the velocity having been much greater in the past, too. Then again, the origin of inertia has always been a bit fuzzy, and this expanding $S^3(r_0)$ geometry could have merits in making it more logical.

Furthermore, the form of Eq. (8) suggests distinguishing the local expansion momentum as a factor in the rest energy,

$$E_0 := c_0(m_0 c) = c_0 p_0 \quad [\mathrm{J} := (\mathrm{m/s})(\mathrm{kg\,m/s})], \tag{9}$$

which is quite astonishing at first sight. In this train of thought, the rest energy can be envisioned to actually consist of momentum $p_0 = m_0 c$, in the only direction orthogonal to the local tangential 3-space, and one can use the global radial expansion hyperspeed c_0, in principle available everywhere, to convert any momenta to energy, in any frame. The conversion to energy happens simply by multiplying the momentum with c_0—in scalar form, as it is not yet known whether this involves some conjugate products or taking the scalar part of some more complex geometrical operation, and in any case one needs to appreciate the spherical symmetry of $S^3(r_0)$ when thinking about the radial velocity c_0 here.

As momentum is a vector quantity, we would need to define the relations in a suitable algebra to be able to manipulate them effectively. But at this point it is fine to just think in terms of vectors and scalars, to not overgeneralize the matters too early.[6] The momentum $p_0 \in \mathbb{G}_4^1$ could be considered as a 4-dimensional vector, orthogonal to all possible ordinary momenta $p_\perp \in \mathbb{G}_4^1$ in the local Euclidean tangential 3-space ("ordinary space"). The perpendicular symbol $\perp$ is meant to be read as a subscript label, not an operator, denoting here the ordinary momenta in the local (possibly tilted) Euclidean tangent space of $S^3(r_0)$. Of course, in suitable coordinates where one basis vector is aligned with the p_0, the ordinary momentum $p_\perp$ can be expressed as a 3-dimensional vector in $\mathbb{G}_3^1$, as the aligned coordinate vanishes, but it may be easier to just work with 4-dimensional vectors directly.

The proposed structure of rest energy in Eqs. (8) and (9) seems perhaps too simple and straightforward, and needs a lot of elaboration to be able to start convincing oneself that it could be consistently built on without some major discrepancies appearing eventually at some point in the process.

[6] I take the liberty of using a sort of mixture of notations here. In some algebras one can easily sum and multiply all kinds of geometrical objects together, but most often those rely on the power of noncommutative operations, which we are not yet using here—all the relations are defined so that one can change the order of multiplications at will. Still we can start appreciating some of the convenient shorthands available, such as $\mathbb{G}_4^1$ which tells that the object in question is in a 4-dimensional space, and that it is a 1-vector, so an ordinary vector instead of an ordered 2-surface or a 3- or 4-volume.

As a first sanity check, the total energy (i.e., rest energy and kinetic energy, but omitting possible internal potential energies), using the 4-dimensional momentum vector $p := p_0 + p_\perp \in \mathbb{G}_4^1$, satisfies then (note the orthogonality of vectors p_0 and $p_\perp$)[7]

$$E_p^2 = (c_0 p)^2 = [c_0(p_0 + p_\perp)]^2 = c_0^2[(m_0 c)^2 + p_\perp^2], \tag{10}$$

which is nice. We have simply multiplied the total momentum p with c_0 here, to get the energy, and raised both sides to the power of two, to reduce the vector quantity to its length squared. If $c = c_0$, the right hand side is exactly equal to the well-known energy–momentum relation most commonly written as

$$E_{\mathrm{rel}}^2 = (m_0 c^2)^2 + (p_\perp c)^2 = m_0^2 c^4 + p_\perp^2 c^2, \tag{11}$$

but now in Eq. (10) it has an interesting and symmetric internal structure, conceivably motivated by an expanding $S^3(r_0)$ geometry in the 4D-hyperspace of the cosmos.

To me this seems quite novel, and throughout the book Suntola shows how different forms of energy (such as radiation, matter, Coulomb energy), can be brought into a sort of standard form $E = c_0 mc$.[8]

B. Radial and tangential mass: Compton and de Broglie

It is widely known that the energy-momentum relation of Eq. (11) can also be described in terms of matter waves:[9]

$$E_{\mathrm{rel}}^2 = (hf)^2 = (\hbar\omega)^2 \propto \left(\frac{m_0 c}{\hbar}\right)^2 + k_\perp^2, \tag{15}$$

where $\hbar := h/2\pi$ is the reduced Planck constant, $\omega := 2\pi f$ is the angular frequency, and $k_\perp := 2\pi/\lambda_\perp$ is the angular wave vector, its magnitude equal to the wave

[7] We are already simulating some of the useful properties of more advanced algebras here: as explained in the appendix, in many algebras the square of a vector is simply its length squared, and also for orthogonal vectors, their sum squared is equal to the sum of their squares, as the cross-terms vanish. We favor those conveniences already here, taking note of orthogonal vectors, even if the mathematical formulations here are not yet totally precise.

[8] Three examples of the similarities in the mathematical appearances of different forms of energy, but without further clarification of the symbols or their meaning at this point:

$$E_0 := c_0 \frac{h_0}{\lambda_0} c = c_0 m_0 c \tag{12}$$

$$E_{\mathrm{rad}} := c_0 N^2 \frac{h_0}{\lambda_\perp} c = c_0 m_\perp c, \tag{13}$$

$$E_{\mathrm{EM}} := c_0 \frac{q_1 q_2 \mu_0}{4\pi d} c = c_0 N^2 \frac{h_0}{2\pi d} \alpha c = c_0 m_{\mathrm{EM}} c. \tag{14}$$

[9] See, e.g., `https://en.wikipedia.org/wiki/Energy-momentum_relation#Matter_waves`, `https://en.wikipedia.org/wiki/Matter_wave`, and the links therein.

number. $\lambda_\perp$ is the wavelength in the 3-space. The angular wave vector $k_\perp$ is by definition orthogonal to the local radial direction, as it is in the tangential 3-space of $S^3(r_0)$, but it is also orthogonal to surfaces of constant phase (wavefronts), pointing in the direction of phase velocity $v_p = \omega/k_\perp = f\lambda_\perp = c/n_p$, where n_p is formally the refractive index. The group velocity $v_g = \partial\omega/\partial k_\perp$, by contrast, is in the direction of wave packet propagation, how the wider envelope shape of the wave's amplitudes—known as the modulation or envelope of the wave—propagates through the 3-space.[10] The group velocity can be thought of as the apparent speed of the peak of a wavepacket, and if the wave is travelling in a linear setting that does not have gain or loss, the group velocity is actually interpreted to be equal to the energy velocity v_e, which is the speed and direction that energy appears to propagate. Energy velocity is also called the signal velocity of the waveform. So there are different apparent directions involved, and one can imagine how rich behaviors can ensue when the conditions can vary (considering also reflections, resonances, and other interactions)—the directions involved are not necessarily always even well defined or meaningful. But usually (in a lossless isotropic medium) the direction of the angular wave vector $k_\perp$ is the same as the direction of wave propagation, and in any case $k_\perp$ is always orthogonal to the wavefront by definition.

Unfortunately, the physical dimensions in the above Eq. (15) do not match, as they are

$$[J]^2 = [(\mathrm{kg\,m^2/s})(1/\mathrm{s})]^2 \neq [(\mathrm{rad/m})]^2 + [(\mathrm{rad/m})]^2. \quad (17)$$

This problem is quite common in theoretical physics, where it is often assumed that one can set $\hbar = c = 1$ and deal with those constants of proportionality later. However, here we cannot do that, as this whole edifice here is premised on keeping track of important factors that could vary. Indeed, it seems as if the whole raison d'être of the model building exercise described here is an aspiration to base the logic on firm foundations using basic concepts, such as mass (*what* in kilograms), distance (*where* in meters), and time (*when* in seconds), so we have to be attentive on all the possibly varying factors, and make sure the definitions are also dimensionally correct.

A more correct formula for the matter waves conceptualized in Eq. (15) would be

$$\begin{aligned} E_\mathrm{m}^2 = (hf)^2 = (\hbar\omega)^2 &= (\hbar c)^2 \left[\left(\frac{m_0 c_0}{\hbar}\right)^2 + k_\perp^2\right] \\ &= (\hbar c)^2 (k_0^2 + k_\perp^2) \\ &= (\hbar k c)^2 \\ &= (c_0 \hbar_0 k c)^2 \\ &= (c_0 m c)^2 \end{aligned} \quad (18)$$

where a 4-dimensional wave vector $k := k_0 + k_\perp \in \mathbb{G}_4^1$ is introduced (and we are again reducing the square of a vector to its length squared). We have also introduced here the intrinsic Planck constant $h_0 := h/c_0$, from where the evolving energy conversion factor c_0 has been removed, and the dimensions of which are thus [kg m]. We will return to this important proposal later, but at this point one can already appreciate that as both Planck constant h and the speed of light c (and c_0) are usually treated as forever constants, we could always do this even in the prevalent paradigm. Assuming that $c \approx c_0$, the current best estimate is thus $h_0 \approx 2.2102 \times 10^{-42}$ kg m.

Now the dimensions in Eq. (18) match properly to energy squared, $[J]^2 = [\mathrm{kg\,m^2/s^2}]^2$. The radial part is equal to the energy-momentum form in Eq. (10) and the resulting total form is correctly the mass-energy equivalent of the total moving mass $m := \hbar_0 k \geq m_0$.

From a purely theoretical perspective, it is rather reasonable that as the expanding $S^3(r_0)$ geometry has separate radial and tangential spaces (one-dimensional and three-dimensional, respectively), the concept of mass itself could have these two components, too. So if the mass $m_0 := \hbar_0 k_0$ is the radial component, then $m_\perp := \hbar_0 k_\perp$ could be the tangential component. But it is not yet clear what is most commonly regarded as the ordinary mass here—in any case we are simplifying a lot here, as the matter wave structures are not explicated yet.[11]

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[10] The envelope wave u_g and the so called carrier wave u_p can be seen already from a superposition of two simple waves w_1 and w_2 (with their respective angular frequencies and angular wave vectors)

$$\begin{aligned} &w_1(x,t) + w_2(x,t) = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t) \\ &= \frac{1}{2}\left(e^{i(k_1 x - \omega_1 t)} + e^{i(\omega_1 t - k_1 x)} + e^{i(k_2 x - \omega_2 t)} + e^{i(\omega_2 t - k_2 x)}\right) \\ &= 2\cos\frac{(k_2 - k_1)x - (\omega_2 - \omega_1)t}{2} \cos\frac{(k_2 + k_1)x - (\omega_2 + \omega_1)t}{2} \\ &= 2\, u_g(x,t)\, u_p(x,t) = u(x,t), \end{aligned} \quad (16)$$

where the resulting wave is a product of two waves: an envelope wave formed by u_g and a higher-frequency carrier wave formed by u_p. The phase velocity of u_g, $(\omega_2 - \omega_1)/(k_2 - k_1)$ becomes the definition of the group velocity in the continuous differential limit.

Adding waves together can easily result in apparent velocities that are very high or very low (or even in the reverse direction)—the situation is similar to a shadow that may appear traveling at a fast speed, but actually the perceived positions are not directly causally related to each other, they are just interpreted as a movement.

[11] Specifically, what is the interpretation of total momentum $h_0 f$ [kg m/s], which is multiplied by c_0 to get the total energy? What is the relation of Compton frequency f (and ω) to an observer riding along with the expansion of $S^3(r_0)$ at the velocity of c_0? How is the local speed of light c used in here? How would one model spatially localized mass structures (for example, a point

From an algebraic viewpoint it is interesting how the proposed intrinsic Planck constant h_0 converts wave vectors to mass, similarly to how c_0 converts momenta to energy.

It seems that Suntola suggests that mass should be viewed abstractly in terms of these kind of oscillating fields, and that taking into account that also radiation has mass (as inferred from its momentum), the abstract mass could then be actually conserved in every kind of interaction, even if it involves conversions between different kinds of masses and energies.

The radial rest mass m_0 can then be conceptualized as the Compton wavelength equivalent of mass, i.e. the radial wave vector is then

$$k_0 := \frac{2\pi}{\lambda_0} = \frac{m_0 c_0}{\hbar} = \frac{m_0}{\hbar_0} \quad \text{[rad/m]}, \tag{19}$$

where the wavelength $\lambda_0 := h_0/m_0$. One kilogram of mass at rest would have the wavelength on the order of 10^{-42} meters, in the radial expansion direction of the $S^3(r_0)$ geometry. For comparison, the proton mass would have the wavelength approximately equal to the range of the strong nuclear force ($\approx 1.3 \times 10^{-15}$ m).

Conversely, the tangential component $m_\perp$ can be seen as de Broglie wave, according to the DU book. Also for light and radiation, $k_0 := 0$ and all the mass is propagating in the local 3-space, as seen in Eq. (13). I have not yet collected the relevant relations here explicitly, but from the above formulations, one can already start to have a feel how this could come together—it seems quite possible to develop these $S^3(r_0)$ inferred concepts of radial and tangential mass further towards more proper comprehension. Thinking mass in terms of wave vectors could clarify the treatments in later sections, but I have not managed to make everything uniform and coherent yet, as this is an evolving position paper reflecting my current imperfect understanding.[12]

For my own purposes, I will include here a rather long five-paragraph quote from Bohm (1996 [1965], pp. 90–92), where one can appreciate how separating these radial and tangential components of the expanding $S^3(r_0)$ geometry could potentially clarify concepts such as "inwardly, reflecting, to-and-fro movement" and "outward displacement through space" (emphasis in the original):

> [...] it will be helpful to begin by introducing a distinction between two kinds of energy. On the one hand, there is the *energy of outward movement* which occurs on the large scale, for example, when a body changes its position or orientation as a whole. On the other hand, there is the *energy of inward movement*, for example, the thermal motions of the constituent molecules, which cancel out on the large scale. It is characteristic of inward movement that it tends to be to-and-fro, oscillating, reflecting back and forth, and so on. (In Einstein's example of the box containing radiation, discussed earlier, the light reflecting back and forth can be regarded as inward movement.)
>
> It is evident that the terms "inward" and "outward" are inherently relational in their meanings. Thus, relative to the large-scale level, molecular movement is "inward," because its over-all outward effects cancel on the large scale. However, relative to the molecular level, it is "outward," because the molecules do undergo a displacement through space that is significant on *this level.* On the other hand, the electronic and nuclear motions are still "inward" relative to the molecular level, although they must be regarded as "outward" when we go to still deeper levels, where *their movements* result in significant space displacements. [...]
>
> In this [$E = mc^2$] connection it must be noted that *every form of energy* (including kinetic as well as potential) contributes in the same way to the mass. However, the "rest energy" of a body has a special meaning, in the sense that even when a body has no visible motion as a whole, it is still undergoing inward movements (as radiant energy, molecular, electronic, nucleonic, and other movements). These inward movements have some "rest energy" E_0 and contribute a corresponding quantity, $m_0 = E_0/c^2$ to the "rest mass." As long as the energy is only "inward," the rest mass remains constant, of course. But as we have seen, internal transformations taking place on the molecular, atomic, and nuclear levels can change some of this to-and-fro, reflecting "inward" movement into other forms of energy whose effects are "outwardly" visible on the large scale. When this happens, the "rest energy" and with it, the "rest mass," undergo a corresponding decrease. But such a change of mass is seen to be not in the least bit mysterious, if we remember that inertial and gravitational masses are merely one aspect of the whole movement, another aspect

mass at rest), as opposed to mere traveling mass planar waves—is Fourier transforming to momentum space the only option? Are point masses some kind of spatially compact resonant structures? Is some kind of chirality or perhaps mass conjugation to a negative frequency possible here? Could CPT-symmetries be mapped to radial, tangential and time components in this kind of an expanding $S^3(r_0)$ model?

[12] For example, the later sections where the radial rest mass is reduced at high velocities, could benefit from reading `https://en.wikipedia.org/wiki/Matter_wave#De_Broglie's_phase_wave_and_periodic_phenomenon`.

> of which is an equivalent energy, exhibited as a capacity to do work on the large scale. In other words, the transformation of "matter" into "energy" is just a change from one form of movement (inwardly, reflecting, to-and-fro) into another form (e.g., outward displacement through space).
>
> It is particularly instructive to consider how, in this point of view, one understands the possibility for objects with zero rest mass to exist, provided that they are moving at the speed of light. For if rest mass is "inner" movement, taking place even when an object is visibly at rest on a certain level, it follows that something without "rest mass" has no such inner movement, and that *all* its movement is outward, in the sense that it is involved in displacement through space. So light (and everything else that travels at the same speed) may be regarded as something that does not have the possibility of being "at rest" on any given level, by virtue of the cancellation of inner "reflecting" movements, because it does not possess any such inner movements. As a result it can exist only in the form of "outward" movement at the speed *c*. [...]
>
> The full development of the point of view outlined [...] concerning the transformation between "rest energy" and other forms of energy, implies that we shall eventually have to understand the so-called "elementary" particles as structures arising in relatively invariant patterns of movement occurring at a still lower level than that of these particles. In such structures even the "rest energy" of an elementary particle would be treated as some kind of "inner," to-and-fro reflecting movement, on a level which is even below that on which nuclear transformations take place."

We do not have time and inclination to analyze this properly yet, but I suspect that Suntola expects that in this expanding $S^3(r_0)$ model, at some point and at some level, this "inward" and "outward" movement will turn out to be related to the proposed radial and tangential components of mass somehow (at least in some capacity, as not everything can be understood as merely thermal motion, and the radial dimension could offer interesting affordances here). But these ponderings can be delegated to future work for now.

C. Electromagnetic radiation in the tangential 3-space

As radiation does not have rest mass (momentum in the local radial direction), its total energy is simply $E_{p_\perp} = c_0 p_\perp = c_0 m_\perp c$ directly, as all the momentum of radiation is in the direction of local tangential 3-space. Suntola argues that the possible superluminal 4-velocities here (as the expansion "lifts" the radiation with it at c_0, too) are illusory and not of great concern, as it is the momentum that matters, and radiation carries momentum only in the propagation direction. He proceeds to assume that radiation actually travels in $S^3(r_0)$ at that local speed c, as defined, with a maximum velocity of c_0, if there are no mass centers present (or if mass is uniformly distributed).

With these very straightforward definitions of an expanding $S^3(r_0)$ in a homogeneous and isotropic setting, where light travels tangentially at velocity $c_0 := \dot{r}_0$ and follows the shape of space (so at cosmological distances, the propagation path of light is a continuous section of an enormous spiral in the hyperspace), Suntola proceeds, in a purely geometrical fashion, to derive rather elegant metrics in relation to most commonly used cosmological distances, measured in redshift (see Table I). Looking at those very simple results, one cannot but wonder that if the metrics cannot be made mathematically any more simpler, still matching the observations, there has to be some relevance in there.

As the standard Riemann metric on $S^3(r_0)$ is $\propto r_0^2$ and the curvature is the inverse, some cosmological observations may become even trivial. Later we will find that in DU, the maximum speed of light,

$$c_0 := \frac{dr_0}{dt} \propto \frac{1}{\sqrt{r_0}}, \tag{20}$$

so the speed of light decreases deterministically with the expansion [see Eq. (23) later]. Most importantly for the model—and most difficult to accept, too—is that the physical clocks, conceivably including even radioactive decay in this theoretical model, progress slower in the same proportion as the radial hyperspeed c_0 decreases, thus obfuscating in the recorded observations the theoretical (but also very real) possibility that things have evolved way faster in the past.[13] On the other hand, from the structure of the proposed rest energy, already

[13] Later we will find that in this model, unit of time, measured as a fixed number of cycles of an atomic clock (so using its frequency f), is getting longer (so slowing down) with the expansion as $f \propto t^{-1/3}$, proportional to the actual time t from singularity. As the expansion of the universe is decelerating in this model, the age of the universe is only two thirds of the current estimate, so 9.3 billion current years, like in Einstein-de Sitter model [see O'Raifeartaigh *et al.* (2015)]. Also due to expansion, the length of a year around the sun is increasing $\propto t$, and due to expansion and tidal friction, the days are also getting longer $\propto t^{1/3}$. It certainly is difficult to keep in mind all the consequences of the

TABLE I The mathematical elegance of the metrics is evident when comparing the models. See Suntola (2018) for derivations and many more illustrative examples.

	DU	GR / FLRW
spatial or comoving distance[ab]	$D_{\text{phys}} := r_0 \ln(1+z) = r_0\theta$	$D_C := r_0 \int_0^z [(1-\Omega_\Lambda)(1+z')^3 + \Omega_\Lambda]^{-1/2} dz'$
optical or light travel distance[ac]	$D := r_0 z/(1+z) = r_0(1-e^{-\theta})$	$D_{\text{LT}} := r_0 \int_0^z (1+z')^{-1} [(1-\Omega_\Lambda)(1+z')^3 + \Omega_\Lambda]^{-1/2} dz'$
angular diameter distance[ad]	using D, $\psi = \theta_d/z$	$D_A := D_C/(1+z)$
luminosity distance[a]	$D_L := D\sqrt{1+z} = r_0 2\sinh(\theta/2)$	$D_L := D_C(1+z)$

[a] The parameter z is the cosmological redshift $\delta\lambda/\lambda$, where λ is any emitted wavelength. Thus, $1+z = \lambda_{\text{observed}}/\lambda_{\text{emitted}}$. In DU, also $z = D/(r_0 - D) = e^\theta - 1$, so $1+z = e^\theta = r_0/(r_0-D)$, where D is the optical distance. The central angle $\theta \geq 0$ is measured along the great circle S^1 connecting the rays from the origin that are unique to the locations being compared. An estimate of the current 4-radius, $r_0 = c_0/H_0 \approx 14$ billion light years, is obtained by taking the Hubble parameter at its present estimate $H_0 \approx 70$ (km/s)/Mpc.

[b] In FLRW, the comoving distance $D_C := D_H \int_0^z E(z')^{-1} dz'$, where $D_H := c_0/H_0 = r_0$ (when interpreted in the DU framework) and $E(z) := [\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_k(1+z)^2 + \Omega_\Lambda]^{1/2}$, where $\Omega_r + \Omega_m + \Omega_k + \Omega_\Lambda = 1$. Most often, but not always, radiation energy density Ω_r and curvature Ω_k are set to zero. Thus the matter energy density $\Omega_m = 1 - \Omega_\Lambda$, in terms of the cosmological constant or "dark energy density" Ω_Λ, resulting in the form in the table. See, for example, Hogg (1999). The integral can be expressed using the hypergeometric function but does not have a simple solution for most parameter values.

[c] Note the following peculiar fact pertaining to the current parameter values in the FLRW cosmology: the light travel distance in the table has a closed form solution $D_{\text{LT}} = r_0(2/3)\,\text{artanh}[(1-\Omega_\Lambda^{1/2} f)/(\Omega_\Lambda^{1/2} - f)]/\Omega_\Lambda^{1/2}$, where $f = [(\Omega_\Lambda^{-1} - 1)(1+z)^3 + 1]^{1/2}$. When looking as far in the past as possible, $D_{\text{LT}}|_{z\to\infty} \simeq r_0(2/3)\,\text{artanh}(\Omega_\Lambda^{1/2})/\Omega_\Lambda^{1/2}$. This attains the value r_0 (so the integral in the original metric reduces to unity) when $\Omega_\Lambda \approx 0.73712$, which also happens to be very close to the parameter value best fitting the supernova observations. Thus, the maximum light travel distance is the Hubble radius $D_{\text{LT}} = r_0 = c_0/H_0 = c_0 T$, which could be interpreted as supporting the same expansion story as the DU. For comparison, the algebraic relations in DU are due to an expanding $S^3(r_0)$ geometry where light and radiation propagate tangentially at the same velocity as the radial expansion proceeds, so there are no parameters other than the current radius and its rate of growth (which are also coupled in the model to the energy density and the gravitation constant, see Eq. 23).

[d] In DU, due to the $S^3(r_0)$ geometry and the expansion of also local gravitationally bound systems (presumably not stars or planets, where thermal pressure or some degeneracy pressure dominates), the angular sizes of distant galaxies and quasars are observed in Euclidean projective geometry $\psi = \theta_d/z$, where z is the redshift observed and θ_d is the angular size of the object in the universal coordinate system with origin in the 4-center of space. In GR, galaxies or planetary systems do not expand, sometimes causing difficulties for defining the various regimes consistently.

one notices that c_0 is an important energy conversion factor, so it is also somewhat expected that it will play a fundamental role in all of the physics in this model.

Suntola reasons that during the expansion, redshift should be interpreted as dilution of the energy density due to an increase in the wavelength of electromagnetic radiation in the direction of propagation. The mass equivalence of a cycle of radiation, $m_\perp = h_0/\lambda_e$, bound to the emission wavelength, is conserved in the lengthened cycle, but the energy density of a cycle of radiation is diluted. Taking into account the overall thinning out of the total energy density due to expansion of the cosmos [later Eq. (23)], the relative energy, however, is conserved.

For those wondering about Maxwell's equations with a variable speed of light, Suntola assumes that magnetic permeability μ_0 is a constant, and the electric permittivity ε_0 is the derived quantity:

$$\varepsilon_0 := \frac{1}{c\mu_0 c_0}, \tag{21}$$

thus allowing the Maxwell's equations still show how fluctuations in electromagnetic fields propagate at a fairly constant speed c in the vacuum solution on the free space.[14]

I am also under impression that Suntola considers gravitational waves in terms of energy propagation, at the speed of light[15]—so that the actual mechanisms of gravity and especially the nature of attractive gravitational potential could still be something entirely different, and likely geometric in character.[16]

proposed model, but they result quite unambiguously from the premises, and they seem to be consistent with observations, provided one can make the mental leap to consider all these details as a coherent whole.

[14] Note that I have not yet studied this in relation to group and phase velocities, and whether $\sqrt{c_0 c}$ is involved. Also $c_0^2 = GM_0/r_0$ of Eq. (23) could be quite suggestive, in relation to Eq. (21).

[15] For example, GW 170817 was a gravitational wave signal observed by the LIGO and Virgo detectors on 17 August 2017, originating from the shell elliptical galaxy NGC 4993. The signal was produced by the last minutes of a binary pair of neutron stars' inspiral process, ending with a merger. It was seen by many observatories across the electromagnetic spectrum.

[16] Even though DU does not speculate a lot about the circumstances near the singularity at $r_0 \approx 0$, it seems to assume that there is a kind of unstructured matter that is the initial form of energized mass in a highly condensed state. During expansion, a share of unstructured matter is converted into visible, structured material, and thus there is some kind of dark matter in the model (as ubiquitous but faint matter fields), perhaps enabling also gravitational energy propagation as mass waves, separately from the actual mechanisms of gravity.

When aiming to actively avoid the spacetime concept, Suntola

Ponderings on the nature of gravity easily veer into the realms of speculative philosophy, which is interesting and important by itself, but next we will see how Suntola has aimed to develop these propositions much further, for instance, by pursuing to understand Mach's principle in light of the expanding $S^3(r_0)$.

D. Gravitational energy in S^3 and Mach's principle

Very general statements of Mach's principle are "mass out there influences inertia here", "local physical laws are determined by the large-scale structure of the universe", or "the reference frame comes from the distribution of matter in the universe".

Mach's principle has been quite difficult to formalize, even though Einstein wrote enthusiastically to Mach himself about a certain class of spherical solutions to the field equations supporting Mach's ideas. For example, Brans and Dicke (1961) assert that the view that "the only meaningful motion of a particle, is motion relative to other matter in the universe, has never found its complete expression in a physical theory." They continue by proposing that:

> From simple dimensional arguments as well as the discussion of Sciama, it has appeared that, with the assumption of validity of Mach's principle, the gravitational constant G is related to the mass distribution in a uniform expanding universe in the following way:
>
> $$\frac{GM}{Rc^2} \approx 1. \tag{22}$$
>
> Here M stands for the finite mass of the visible (i.e., causally related) universe, and R stands for the radius of the boundary of the visible universe. [...] This relation has significance in a rough order-of-magnitude manner only, but it suggests that either the ratio of M to R should be fixed by the theory, or alternatively that the gravitational constant observed locally should be variable and determined by the mass distribution about the point in question.

They also ponder that

> gravitation provides another characteristic mass [than the mass of electron], $(\hbar c/G)^{1/2} \approx 2.16\times10^{-8}$ kg, and the mass ratio, the dimensionless number $m_e(G/\hbar c)^{1/2} \approx 5 \times 10^{-23}$, provides an unambiguous measure of the mass of an electron which can be compared at different spacetime points. [And that] the odd size of this dimensionless number has often been noticed as well as its apparent relation to the large dimensionless numbers of astrophysics. The apparent relation of the square of the reciprocal of this number [...] to the age of the universe expressed as a dimensionless number in atomic time units and the square root of the mass of the visible portion of the universe expressed in proton mass units suggested to Dirac a causal connection that would lead to the value of [the mass ratio] changing with time. [...] Dirac postulated a detailed cosmological model based on these numerical coincidences. This has been criticized on the grounds that it goes well beyond the empirical data upon which it is based.

Now, I am not really familiar with Brans-Dicke models—also called scalar-tensor theories—and their various challenges, and have not made a proper literature search on Mach, Sciama, Dirac, and where this line of questioning is currently at. For some original papers, see aforementioned (Brans and Dicke, 1961), also (Dicke, 1962a,b), the latter of which contain suggestive ideas such as "The extremely small value of the gravitational coupling constant [...] is then recognized as the effect of the enormous amount of matter in the universe generating a scalar field which acts to depress the value of m_e, the mass of an elementary particle." See also Hoyle and Narlikar (1964). I do not know how the proposed scalar field and the famous Higgs field are related, for example [see Kaiser (2007) for discussion on these connections].

Still, very interestingly, Suntola has clearly thought deeply about very similar ideas in relation to Mach, and with the research direction informed by the expanding $S^3(r_0)$ geometry, he proceeds to formulate these ideas quantitatively in the most direct way. He simply assumes that inertia is related to the work done in reducing the radial rest mass of an object moving in the tangential 3-space—that there is a gravitational effect of the totality

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does not use retarded potentials, but assumes that the location of the potential follows the changes of the source immediately, à la Laplace—which will be a dealbreaker for many, I assume. For a lively discussion on the subject of action-at-distance, see Hesse (2005 [1961]). But then again, if gravity was essentially based on the geometric character of the expanding $S^3(r_0)$, there is no apriori reason to dismiss this idea as delusional. Indeed, if somebody would try to wield the presumed geometrical quality of almost instant gravitational potential for superluminal communication, it would certainly prove out to be quite difficult, as Newtonian gravitational potential $1/r$ fades away very quickly in every direction, and the resulting forces as gradients would be even more feeble, also taking account the inertia of the responses. Even major changes would be drowned out in a sort of Newtonian noise anyway, as all masses are in constant flux everywhere. For example, in cases where gravitational waves have been detected, the energy from which the gravitational potential originates, has still been located at the approximate vicinity (on the astronomical scale) of the event for a long time before and after the event—so there is not a lot to observe in terms of possible instant gravitational potential changes at a distance.

of mass in space on the object in motion (and at rest, too), and that one can feel that as inertia. If one can entertain the idea of classical Newtonian potentials, it is not so far-fetched to consider the aggregate operating at almost infinite distances, however weak (and there is geometry involved, too).

With the assumption that gravity and radial rest energy are actually the same thing (sort of conjugates to each other), Suntola formulates a Machian balance equation in an expanding, homogeneous $S^3(r_0)$ (where $c = c_0 = \dot{r}_0$), as

$$E_{m_0} := c_0 m_0 c_0 = \frac{m_0 G M_0}{r_0}, \tag{23}$$

which turns out to be the key to most of the dynamics in the model. The radial rest energy $m_0 c_0^2$ of test mass m_0 and the gravitational energy $GM_0 m_0/r_0$ from the whole of space M are postulated to be in equilibrium, so that their difference as total energy is actually zero. The gravitational energy from the rest of space works as a Machian dual complement to each point mass in question.

Importantly, it is the speed of light c_0 and the expanding radius r_0 (which is also the radius of curvature here) that are elastic, not the Newtonian constant of gravitation G, or total mass M, in this Eq. (23).[17] It is indeed the total mass M of the universe that is conserved in various abstract forms throughout the expansion—the energy E_{m_0} of motion and gravitation will diminish over time as r_0 grows, but the total energy will stay zero, in this model. Also note that Eq. (22) of Sciama and Brans-Dicke, and Eq. (23) of Suntola, are essentially identical, but the latter perhaps benefits from a more plausible geometric construction, giving it structure and meaning.

With guidance provided by the $S^3(r_0)$ geometry, Suntola then assumes that one can integrate the scalar gravitational potential directly inside the $S^3(r_0)$ manifold for the whole of space, without using the Haar measure for force field integration or some such techniques. M_0 is then the apparent mass of the universe, having a kind of a virtual effect from the radial zero on any mass m_0 due to the curvature of space, estimated by integrating the $1/r$ gravitational potential only tangentially inside the $S^3(r_0)$ manifold to arrive at a correction factor of

$$M_0 := M \int_0^{\pi} \frac{2\sin^2\theta}{\pi\theta} d\theta \approx 0.776\, M, \tag{24}$$

where it remains to be seen if the integral is to be modified in future work, as the $S^3(r_0)$ space is naturally periodic (circular).[18] See Suntola (2018, pp. 87–90) for some details. Integrating the Newtonian $1/r$ gravitational potential over the whole of S^3 space can be equivalently characterized as some kind of tension or negative pressure along the volumetric surface of the $S^3(r_0)$ manifold, but the resulting sum direction is still clearly towards the mathematical origin of the 4D-hyperspace.

It is quite incredible if such simple relations could model the dynamics of the cosmos at large, so it would require, again, very serious elaboration and empirical validation to even begin being convinced about the potential realities here. On the other hand, things ought to be as simple as possible on the large scale, and these kind of almost linear relations have proven out to be immensely useful in many branches of science, so it is perhaps not as extraordinary hypothesis, either.

With that in mind, using only the identities in Eqs. (23) and (24), and the well-known surface area volume $S_V = 2\pi^2 r_0^3$ of the $S^3(r_0)$ manifold, one can then estimate the present day mass density $\rho_0 \approx M/S_V$ of the universe in this model,

$$\rho_0 \approx \frac{r_0 c_0^2}{0.776\, G S_V} = \frac{H_0^2}{1.552\, G\pi^2} = 5 \times 10^{-27}\ \text{kg/m}^3, \tag{25}$$

where an estimate of the current 4-radius

$$r_0 = \frac{c_0}{H_0} \approx 1.4 \times 10^{26}\ \text{m} = 14\ \text{billion ly}, \tag{26}$$

has been used. H_0 is the Hubble parameter, that tells how fast galaxies are moving away from Earth proportional to their distance, here taken at its present estimate ≈ 70 (km/s)/Mpc. When interpreted in the expanding $S^3(r_0)$ geometry, H_0 is simply a geometric relation between radial velocity and the apparent linear velocity based on spatial distance, as $H_0 := \dot{D}_{\text{phys}}/D_{\text{phys}} = [d(r_0\theta)/dt]/r_0\theta = c_0/r_0$ (recall the metrics in Table I).

Surprisingly, the calculated mass density ρ_0 of the universe in Eq. (25) is in agreement with the current estimates in the same order of magnitude, even though the models are likely based on very different assumptions about the nature of space, gravity, radiation, mass, and time, among other differences. Of course, many of

[17] The current best estimate is $G = 6.674\,30(15) \times 10^{-11}\ \text{m}^3/\,\text{kg}\,\text{s}^2$.

[18] Even more speculatively, it is interesting that in a closed, circular space such as $S^3(r_0)$, there is a mathematical possibility of essentially instant but very faint gravitational interaction among itself "the other way around", from multiple directions and hypothetically even multiple times, creating resonance effects. Whether this relates in any way to current practices in mathematical physics, of preparing the classical fields with Fourier transforms and related operations for quantization, I really do not know, but felt compelled to note this interesting fact. The integral in Eq. (24) is almost one when integrating over the whole circumference to $\theta = 2\pi$, but slowly diverges for larger values of θ. If there would be an extra $1/\theta$ factor inside the integral, it would converge to exactly unity when the limit is taken to infinity. Also the density approximation in Eq. (25) would then be $\rho_0 \approx c_0^2/2\pi^2 r_0^2 G = 3.6 \times 10^{-27}\ \text{kg/m}^3$.

the current models are much more specific in their parameters, and "the same order of magnitude" (around 10^{-27} kg/m^3) may not impress the expert, but still, the prediction of mass density could easily be all over the place with this construction, and it clearly is not.

The conserved mass M of the universe is then

$$M \approx \frac{r_0 c_0^2}{0.776\, G} = \frac{c_0^3}{0.776\, G H_0} \approx 2.4 \times 10^{53} \text{ kg}. \tag{27}$$

There is a lot more to be said in terms of comparisons with current estimates of the material, spatial, and temporal dimensions of the cosmos. The reader is referred to Suntola (2018, 2021) for various attempts, most rather successful, to intepret the existing cosmological records under this model.

Treating the foundational balance equation Eq. (23) as a differential equation (as $c_0 := \dot{r}_0$), one can quite easily study the expansion of this gravitating $S^3(r_0)$ model in a closed form by integration, resulting in

$$r_0 = \left(\frac{3}{2}\sqrt{GM_0}\, t\right)^{2/3}, \tag{28}$$

where t is the time elapsed since singularity. The unique feature here—also mentioned earlier and described later—is that experienced time, as measured by atomic clocks, is instead postulated to slow down in the same proportion as the speed of radial expansion of the $S^3(r_0)$ manifold decreases. The time derivative of Eq. (28) relates then the speed of light c_0, but also that experienced rate of progress, to the sort of universal time t from singularity:

$$c_0 := \frac{dr_0}{dt} = \left(\frac{2}{3}\frac{GM_0}{t}\right)^{1/3}. \tag{29}$$

Even though the nature of singularity at $t \approx 0$, where r_0 approximately vanishes, is not theorized much here, the mathematical form of Eq. (29) suggests that the speed of light has been there essentially infinite, for an infinitesimally short duration.[19]

As light travels following the shape of space, and at the postulated velocity c_0 [in this expanding, homogeneous $S^3(r_0)$ space], we can calculate the angular frequency ω_0 (the speed of light as measured in radians per second, from the point of view of the center of the hyperspace) using Eqs. (28) and (29), arriving at a rather nice relation

$$\omega_0 := \frac{c_0}{r_0} = H_0 = \frac{2}{3\,t}. \tag{31}$$

The angular speed of light ω_0 (which happens to equal Hubble parameter H_0 in this model) decreases in inverse proportion to the elapsed time t from singularity.[20] Also comparing to Eq. (25), the overall mass density is diluted as $\rho_0 \propto t^{-2}$.

Integrating Eq. (31) between two moments of time, t_0 and t, we get

$$\theta_t = \int_{t_0}^{t} \omega_0\, dt' = \int_{t_0}^{t} \frac{2}{3\,t'} dt' = \frac{2}{3} \ln \frac{t}{t_0}, \tag{32}$$

where the resulting θ_t is the central angle covered moving at the speed of light c_0 for a duration of $\Delta t = t - t_0$. The angle has covered a proportion P of the full circle (or spiral, actually) around the expanding $S^3(r_0)$ manifold when $\theta_t = 2\pi P$, that is when

$$t = e^{3\pi P} t_0. \tag{33}$$

So interestingly, even though the speed of light c_0 is decreasing, limiting the maximum traveling speed, so is the radial expansion velocity decreasing—in such a proportion that light could reach the same 3-space location in the expanding $S^3(r_0)$ infinitely many times, in the long run. Each such roundtrip takes $\exp(3\pi) \approx 12\,392 \approx 111^2$ times longer, however, than the previous one. Also at each midpoint, at $t/t_0 = \exp(3\pi/2) \approx 111$, the light would reach the unique antipodal point at the other side of the $S^3(r_0)$ manifold (with homogeneous space, where masses would not deflect the light from the great circle), which is an interesting idea.

These calculations do not perhaps mean much at the present day, but could prove out to be useful for those that consider the times nearer the singularity where the cosmic background radiation originates, where the velocities are much greater and distances much smaller. Also in logarithmic scale (or exponential time) the relations are linear.

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[19] Also the form of the Eq. (23) could quite obviously support the idea that there would have been some sort of existence already before the singularity, as a sort of collapsing process, as solving for c_0 we get two solutions,

$$\frac{dr_0}{dt} := c_0 = \pm\sqrt{\frac{GM_0}{r_0}}, \tag{30}$$

suggesting an era of radial shrinking as a possibility. The energy buildup in such a crash would now be conserved in the expansion phase—like an eternal spherical pendulum, from infinity to infinity. But for the faint-hearted, embracing these kind of hypothetical inferences about the origins so long ago is not a prerequisite for day-to-day model development and use.

[20] Note that using radians here is somewhat arbitrary, as from a geometric point of view, it is often better to consider angles in terms of areas of circular sectors on a unit disk (as areas are equivalent both in Euclidean and hyperbolic geometry, but not distances, and then full angle is simply π, as the area of the unit disk). See Sobczyk (2019) for motivations. By using that kind of sector area angles instead, the above relation in Eq. (31) would then read $\omega_A = \omega_0/2 = 1/3t$, resulting in different constants of proportionality down the road—without changing the meaning of the relations, of course.

This Machian analysis of the structure of rest energy as expansion movement in a cosmic gravitational equilibrium can serve as the global view, from where we will again descent towards the mass centers—next we will study how Suntola combines the movement of space to the movement of matter in $S^3(r_0)$, resulting in a more local treatment of the matters at hand.

E. Kinetic energy and Mach's principle refined

Recall that the proposed structure of rest energy in Eqs. (8) and (9), when combined with movement in the tangential 3-space, results in the well-known energy-momentum relation in Eq. (10). So in this model, one can translate from directional momenta to scalar energies by multiplying the momenta with radial expansion hyperspeed c_0, and from scalar energies to directional momenta by dividing accordingly, provided one keeps track of the vector components. As usual in relativistic physics, the kinetic energy E_K is then to be understood as the energy increase due to spatial momentum $p_\perp$, i.e., as the difference between the resulting total energy and the rest energy. It can be conceptualized as the mass-energy equivalent

$$E_K := E_p - E_0 = c_0 m_\Delta c \geq 0 \tag{34}$$

of the mass increase m_Δ, as compared to the rest mass m_0. Recall that $p = p_0 + p_\perp$, and note that with these definitions, the invariant rest mass in $p_0 = m_0 c$ is not affected. The conceptualized mass increase m_Δ affects only the resultant momentum vector p and its spatially projected component $p_\perp$, the same way, so collectively:

$$p_0 = m_0 c \tag{35}$$
$$p_\perp = (m_0 + m_\Delta)\, v = \gamma m_0 v \tag{36}$$
$$p = (m_0 + m_\Delta)\, c = \gamma m_0 c, \tag{37}$$

where v is the ordinary 3-velocity in the tangential space of $S^3(r_0)$, and γ is a proportionality factor $(m_0 + m_\Delta)/m_0$, to be utilized soon. Note that Eq. (37) results simply from the mass-energy equivalence of total energy, as $p = E_p/c_0$.

Suntola reasons that following Mach's principle, there must exist a kind of a dual to this kinetic energy increase in the tangential space of $S^3(r_0)$, visible in Eq. (36). This has certainly been a culmination point in the model development; he identifies the implicit, missing element as the global gravitational energy decrease in the local radial direction.

For Suntola, gravitation has features of global gravity (as manifested in radial rest mass), and local gravity (as manifested in local tangential space and movement). The radial expansion direction of the $S^3(r_0)$ geometry, which the proposed structure of rest energy is based on, is paramount, as that is the fresh idea here.

To begin separating these components, we can first look into how mass and gravity are traditionally conceptualized:[21]

> Physics has two concepts of mass, the gravitational mass and the inertial mass. The gravitational mass is the quantity that determines the strength of the gravitational field generated by an object, as well as the gravitational force acting on the object when it is immersed in a gravitational field produced by other bodies. The inertial mass, on the other hand, quantifies how much an object accelerates if a given force is applied to it. The mass-energy equivalence in special relativity refers to the inertial mass. However, already in the context of Newton gravity, the weak equivalence principle is postulated: the gravitational and the inertial mass of every object are the same. Thus, the mass-energy equivalence, combined with the weak equivalence principle, results in the prediction that all forms of energy contribute to the gravitational field generated by an object. This observation is one of the pillars of the general theory of relativity. —Wikipedia

Tentatively, as far as I can see, Suntola is in agreement that these concepts of the gravitational mass and the inertial mass are still equal (which is a relief!), but only if the masses in the above quote means the ordinary, tangential mass.[22] By looking at the resulting model from a Machian perspective, he is convinced that it is the concept of radial rest mass that is in need of augmentation, to account for global gravitational effects. Specifically, the mass $m_\perp := m_0 + m_\Delta$ in Eqs. (36) and (37), is the normal tangential relativistic mass that we are used to (usually defined via the total energy, as $m_\perp = E_p/c_0 c$), that contributes also to the gravitational field generated by an object, as stated. But that the radial mass m_0 in Eq. (35), or at least its mass-energy equivalence as global gravitational energy, needs further work.

By studying visual diagrams of $p = p_0 + p_\perp$ (where p_0 and $p_\perp$ are orthogonal by definition) that exhibit energy and momentum conservation in equilibrium, Suntola finds that with these definitions, one can derive the Lorenz-factor γ and its inverse $\alpha := 1/\gamma$ purely geometrically, utilizing ordinary trigonometry directly. One finds

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[21] The quote exhibits common ideas from `https://en.wikipedia.org/wiki/Mass-energy_equivalence#Relation_to_gravity`

[22] Suntola has hesitations related to the equivalence principle and the Schwarzschild solutions, among some other reservations (Suntola, personal communications).

that by studying the angle φ_m between p_0 and p,

$$\begin{aligned}\cos\varphi_m &= \sqrt{1-\sin^2\varphi_m}\\ &= \sqrt{1-\left(\frac{p_\perp}{p}\right)^2}\\ &= \sqrt{1-\left(\frac{\gamma\, m_0 v}{\gamma\, m_0 c}\right)^2}\\ &= \sqrt{1-\frac{v^2}{c^2}},\end{aligned} \tag{38}$$

and on the other hand,

$$\cos\varphi_m = \frac{p_0}{p} = \frac{m_0 c}{\gamma\, m_0 c} = \frac{1}{\gamma} = \alpha. \tag{39}$$

Combining the above Eqs. (38) and (39), we arrive at the familiar form of the Lorentz-factor

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{40}$$

which means that $\gamma = 1/\cos\varphi_m = \sec\varphi_m$, determined by the accumulated angle φ_m from p_0 to p. All this is done without hyperbolic trigonometry, diverging in a major way from usual treatments of relativistic physics.[23]

I mention the secant here explicitly, as in addition to using the algebraic identity $1/\cos = \sec$, one can see that by using the magnitude of p_0 to define a circle in the cross-section of the hyperspace where both p_0 and p lie, one can inspect that the magnitude of the total momentum p must be the secant of the angle φ_m between the p_0 and p, multiplied by the magnitude of the p_0, and that similar constructions are apparent in stereographic projections from higher dimensional spaces to lower, utilized so effectively in geometric formulations of quantum mechanics [see, for example, Sobczyk (2019, p. 142)].[24]

Now, having derived the famous γ-factor of special relativity from the constraints of an expanding $S^3(r_0)$ geometry, as shown above, Suntola assumes that as the gravitational mass and the inertial mass are the same concept, and increase equivalently with velocity, as clearly

$$m_\perp = \gamma m_0 \tag{42}$$

in Eq. (36), ordinary mass $m_\perp$ is actually conceptually related to the tangential 3-space of the expanding $S^3(r_0)$, and therefore there must exist symmetrically a sort of dual space of a kind of a radially reduced rest mass concept

$$\bar{m}_0 := m_0/\gamma = m_0\alpha = m_0\cos\varphi_m = m_1, \tag{43}$$

where, in addition to Eq. (39), I have utilized an overbar notation to separate this radially reduced rest mass $\bar{m}_0$ from the invariant (in this energy frame) rest mass m_0.[25]

Suntola claims that based on his analysis, this decrease in radial rest mass $\bar{m}_0$ in Eq. (43) is observed as a decreased frequency of atomic clocks, and also as increased Compton wavelength of matter. It can be calculated utilizing the inverse of the γ, or rather the $\cos\varphi_m$ directly, and he claims it is exactly equal to the global gravitational energy decrease, in the local radial direction, due to the increased momentum in the tangential 3-space. It makes sense that then there is a sort of multiplicative balance $m_\perp\bar{m}_0 = m_0^2$, and evidently this reduction of global gravitational energy in the local radial direction suggests a rather vivid image of surfing the expanding S^3 tangentially as velocity increases, thus getting lighter (in terms of radially reduced rest mass $\bar{m}_0$ only). This is also analyzed later in the DU book in terms of centripetal and centrifugal forces in the 4D-hyperspace.

In other words, Suntola proposes that the reduced radial rest masses $\bar{m}_0$ (and those related $\cos\varphi_m$ factors) are crucially important in understanding how kinematics affect energies in nested energy frames, eventually affecting how even time is measured. One can already anticipate that there is a possibility to bring in kinematic time dilation effects as the γ-factor arises rather naturally here, and we will return to these important ideas at the end.

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[23] Looking at this very simple derivation, I find it still a bit hard to believe—by its syntactical correctness it is undeniably valid, of course, and due to its geometric construction, the meaning of γm_0 in Eqs. (36) and (37) is now clearer than ever. But in terms of physics as a scientific inquiry, I am quite baffled at what is going on in here, to be honest. How has this potentially fundamental shortcut somehow escaped so many theorists of mathematical physics is beyond me. Is this somehow in the dark from usual methods and ways of thinking, which are based on derivatives and integrals? How are the velocity addition formulas looking like? Are there some major roadblocks ahead in this line of investigation that I am not yet seeing here?

[24] Explicitly, following Sobczyk (2019, p. 142), the basic equation governing stereographic projection of the unit 3-sphere S^3 from the point $-\mathbf{e}_0 \in \mathbb{R}^4$ onto the flat xyz-space of $\mathbb{R}^3$, is

$$\mathbf{m} = \left(\frac{\hat{\mathbf{a}}+\mathbf{e}_0}{2}\right)^{-1} = \mathbf{x}_m + \mathbf{e}_0, \tag{41}$$

where $\hat{\mathbf{a}} \in S^3$ and $\mathbf{x}_m \in \mathbb{R}^3$. It implies that $\mathbf{m}\cdot\mathbf{e}_0 = 1$, and by inspection, $|\mathbf{m}|$ is the secant, so the inverse of cosine is involved in inversion of that mean of $\hat{\mathbf{a}}$ and $\mathbf{e}_0$ to outside of the unit sphere, to the 3-volume orthogonal to $\mathbf{e}_0$. This is just a detached note at this point, though.

[25] I must admit that it has been quite difficult to get into the roots of this distinction between m_0 and $\bar{m}_0$, as clearly one can modify the m_0 at will, as everything in Eqs. (34–40) is linearly proportional to it. The relations are equally valid for any rest mass, so one must choose some m_0 as a starting point, otherwise the kinetic mass increase in $m_\perp$ is meaningless. And then one must use some m in calculations in any case, and clearly most often it is the ordinary increased tangential mass $m_\perp$, as defined. So currently I am under impression that $\bar{m}_0$ is not so much about the moving matter in this energy frame, but it is about the effective, radially reduced rest mass as a reference for the subframe to use. More about these important concepts later.

To start completing this already rather advanced Machian analysis of gravity in an expanding $S^3(r_0)$ geometry, recall that already in Eqs. (8) and (9), a bilinear dependency to both radial rest mass m_0 and speed of light c in the structure of rest energy was suggested. Suntola observes that formally also the speed of light c could then vary locally, and this new degree of freedom in the model is promptly developed to the extremes with a great impact.

For example, this results in a very curiously symmetric structure for the updated Eq. (34), as now taking into account the free fall in a gravitational potential, the total kinetic energy

$$E_K = c_0(m_\Delta c - c_\Delta m_0), \tag{44}$$

where $m_\Delta \geq 0$ is the tangential mass increase released by the accelerating system, due to the increased tangential momentum in a constant gravitational potential, and now $c_\Delta \leq 0$ is the decrease in the local speed of light due to the local, scalar gravitational potential, tilting the local 3-space in the expanding $S^3(r_0)$ manifold towards the mass centers, resulting in increase in kinetic energy in free fall.

All this, if true, could be fairly significant, as we have come a long way from the venerated $E = m_0c^2$, where rest mass m_0 and speed of light c are invariant, to this fascinating Eq. (44), where these Machian concepts of inertial differences are in this antisymmetric juxtaposition. While still being bilinear in m_0 and c, it reduces to traditional mass equivalent kinetic energy if $c_0 = c$ and $c_\Delta = 0$, thus making them formally perfectly compatible.

Thus, it seems that to make the $S^3(r_0)$ model consistent with a wide range of observations, with a small amount of premises, Suntola proposes that gravity is actually multiplicatively separable. Firstly, to the effect of motion in the local tangential space, where gaining kinetic energy actually decreases global gravitational energy in the local radial direction by γ^{-1} (but increases ordinary tangential gravitational mass by γ, as usual), resulting in reduced radial rest mass $\bar{m}_0$, and secondly, to the effect of gravitational potential in the local tangential space, affecting local speed of light c and wave propagation, by tilting the expanding 3-space locally in the hyperspace, to conserve the global gravitational balance—all this inspired and in the spirit of Mach's principle.

Both are fairly wild ideas, as the invariance of rest mass and the constancy of speed of light are so deeply ingrained ideas in the sciences. Suntola considers that either (1) c and m_0 are constant, therefore the time and distance need to be flexible, or (2) time and distance are rigid measures, and now c and m_0 can vary. He declares that the first assumption leads to kinematics and metrics based models, such as general relativity, and the latter to system-oriented total energy based models [that he calls zero-energy principle models, in reference to Eq. (23)]. Evidently DU explores the latter eventuality.

F. Nested energy frames

As implied earlier, due to practical needs of computation, the model discretizes or layerizes the whole of space as nested energy frames in terms of motion and gravitation. As the expanding $S^3(r_0)$ geometry is well defined, there is a sort of homogeneous, perfectly symmetrical view of the cosmos at the top, from where the conceptual frames descend, layer by layer, towards the mass centers under scrutiny.

It should come as no surprise that this approach differs from many other models in physics, where typically one tries to banish all absolute coordinates whatsoever, informed by the relativity principle, and aims for a covariant field theory, or perhaps starts by aggregating mass centers, bottom-up. Here everything is related to the whole of cosmos top-down, which in this model has to be at least somewhat absolute, as the expanding $S^3(r_0)$ as a mathematical object in a rigid 4-dimensional Euclidean hyperspace necessarily dictates some absolute definitions, such as the 4-radius r_0, or some specific locations in the 3-space manifold to exist, to make sense.

In this model, the whole is related both globally and locally to mass centers by aiming to quantify the Mach's principle, as shown already in Eqs. (23), (43), (44), and in various other occasions where the proposed structure of rest energy was discussed. So in a sense, relativity as a concept is still important in this model, but now in relation to the whole of cosmos (as a background), and in relation to the point of view of the observer and the observed, specifically to their local, physical states—as the energy frames under discussion here affect the progression and measurement of time in the frame, for example.[26]

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[26] Compare to Bohm (1996 [1965], p. 137) (emphasis in the original):

> "It can be seen that all these [relativistic] considerations arise out of the need to take into account the important fact that *the observer is part of the universe.* He does not stand outside of space and time, and the laws of physics, but rather he has at each moment a definite place in the total process of the universe, and must be related to this process by the same laws that he is trying to study. As a result, because of the very form of these laws of physics, which imply that no physical action can be transmitted faster than light, there are certain limitations on what can be known by such an observer at a given moment.
>
> In the quantum theory the consequences of the fact that the observer is part of the universe are even more striking. For when one takes into account the indivisible quanta of action which connect the observer with what he observes, one sees that every act of observation brings about an irreducible *participation* of the observer in what he observes, a participation which entails a *disturbance* of the observed system. As a result, there is, as Heisenberg showed in his discussion of the indeterminacy principle, a minimum uncertainty in the accuracy of every kind of measurement. But it is perhaps not so generally realized that the relativity theory by itself leads to the necessity of a sort of inherent uncertainty in our predictions, different from that which

In various parts of the DU book Suntola addresses the apparent problem of the aether, the question of which arises very naturally in an expanding $S^3(r_0)$ model, even if it were directly unobservable (as it could have effects via the radial rest mass and local speed of light, for example). Note that I have not reviewed all the various tests of relativity (and perhaps never will), such as Trouton-Rankine, Kennedy-Thorndike, Ives-Stilwell, Michelson-Morley, Sagnac effect, Shapiro time delay, among others, many of which Suntola addresses, for example in (Suntola, 2018, pp. 43–49). For the time being, I just assume that as a mathematical model, there is a place for discussions about a model and its formal predictions, and about design and interpretation of experiments. As the expanding $S^3(r_0)$ geometry is so well defined and intellectually quite pleasing, I would not be too surprised if the "aether" as a universal coordinate system would actually exist, just masked by some kind of frame dragging effects[27]—but getting into the roots of the various experiments done so far may distract from my own goals at this point, which is scientific communications about the model in question, to be able to make progress here.

So concretely, Suntola assumes that the cosmos can be modeled as nested energy frames, such as galaxies, star systems, planets, and vehicles. The cascade of frames conceptually descend towards mass centers from a perfectly symmetrical, expanding $S^3(r_0)$ in an absolute 4-dimensional hyperspace, and the frames are multiplicative in nature.[28]

Ignoring the apparent complications for defining the frame boundaries, we can continue the analysis of the composition of the frames themselves.

1. Factorized structure for rest energy

Specifically, assuming Eq. (43) and following Suntola (2018), if we examine the effect of relative velocity v_i (as compared to its parent frame) of each frame i on reduced radial rest mass m_n on the n:th frame, we could write a more developed, factorized version of Eq. (8), the proposed structure of rest energy, as

$$\begin{aligned} E_n &= c_0 \left(m_0 \prod_{i=0}^{n-1} \sqrt{1 - \frac{v_i^2}{c^2}} \right) c \\ &= c_0 \left(m_0 \prod_{i=0}^{n-1} \sqrt{1 - \beta_i^2} \right) c \\ &= c_0 \left(m_0 \prod_{i=0}^{n-1} \cos \varphi_i \right) c \\ &= c_0 \left(m_0 \prod_{i=0}^{n-1} \alpha_i \right) c \\ &= c_0 \bar{m}_{n-1} c \\ &= c_0 m_n c, \end{aligned} \tag{46}$$

where several equivalent forms found in the work are displayed. v_i is the relative velocity of frame i in relation to the parent frame $i-1$, affecting the subframe $i+1$ and its subframes below. Then c is the local, absolute speed of light, here assumed at first the same in all frames i. Note that m_0 is the invariant rest mass here, it is just reduced through the nested energy frames, resulting in reduced radial rest mass $m_n = \bar{m}_{n-1} = m_{n-1}\alpha_{n-1}$.

There is a serious danger of making off-by-one index mistakes here, as one can appreciate the difficulty in keeping track of whether a particular symbol relates to the frame i or its parent frame $i-1$, or perhaps to its subframe $i+1$. But at this point one can also appreciate the possibilities for mathematical generalization here.

Then, symmetrically, we could write a similar factorized version in regard to the local, reduced speed of light c—without considering the actual meaning of the symbols much yet—resulting in the proposed factorized ver-

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follows from quantum theory, and yet not entirely dissimilar in its implications."

[27] For example, see (Suntola, 2018, p. 47) and its references for discussion on interpreting the Michelson-Morley experiment. I regret that I have not yet summarized it here.

[28] As there are no strict specifications how the discretization to energy frames is done—are they based on some centers of mass whose internal potential energies are somehow important enough, exceeding some analytical thresholds in their nested environments, or what—the division should be mostly epistemological, not necessarily ontological. But this is not strictly true, as one can skip a frame quite easily, intentionally or accidentally, and it should not affect the results, but as far as I see, it affects.

I should study DU book 4.1.4 "The system of nested energy frames" to be able to make progress here. At this preliminary point, I simply considered two frames, with relative speeds v_1 and v_2, where the latter is relative to the first. Then the actual speed $v_2' = v_1 + v_2$ (omitting the complications of different velocity orientations). Comparing the aggregate effect of two frames v_1 and v_2, and the effect of skipping the frame v_1 directly to frame v_2', the reduction effect on radial rest mass,

$$\sqrt{1 - \frac{v_1^2}{c^2}} \sqrt{1 - \frac{v_2^2}{c^2}} \neq \sqrt{1 - \frac{(v_1+v_2)^2}{c^2}}, \tag{45}$$

which violates the idea of arbitrary skipping of frames, making them necessarily ontological, in this model. Actually Suntola is of the opinion that these kind of kinematic analyses are not relevant here—one should rather study the building up of kinetic energy, and there the radial and tangential dimensions are both present at the same time, making the analysis more relevant.

There are still many crucial elements in this vast work that I unfortunately have not understood well yet at this early point. In any case, it seems to be an open question how the frames should be defined analytically, preferably from just the mass densities in motion.

sion of rest energy

$$\begin{aligned} E_n &= c_0 m_0 \left[c_0 \prod_{i=0}^{n-1} \left(1 - \frac{GM_i}{r_i c_i c_0} \right) \right] \\ &= c_0 m_0 \left[c_0 \prod_{i=0}^{n-1} (1 - \delta_i) \right] \\ &= c_0 m_0 \left(c_0 \prod_{i=0}^{n-1} \cos\phi_i \right) \\ &= c_0 m_0 \left(c_0 \prod_{i=0}^{n-1} \eta_i \right) \\ &= c_0 m_0 c_n, \end{aligned} \tag{47}$$

where we have assumed that now the radial rest mass m_0 is constant, but now speed of light c_0 is reduced by η_i (eta) factors through the nested energy frames, arriving at the local speed of light $c := c_n$.

Combining the above Eqs. (46) and (47), we finally get a relatively concise equation for the factorized structure of rest energy

$$\begin{aligned} E_n &= c_0 \left(m_0 \prod_{i=0}^{n-1} \sqrt{1 - \frac{v_i^2}{c_i^2}} \right) \left[c_0 \prod_{i=0}^{n-1} \left(1 - \frac{GM_i}{r_i c_i c_0} \right) \right] \\ &= c_0 \left(m_0 \prod_{i=0}^{n-1} \sqrt{1 - \beta_i^2} \right) \left[c_0 \prod_{i=0}^{n-1} (1 - \delta_i) \right] \\ &= c_0 \left(m_0 \prod_{i=0}^{n-1} \cos\varphi_i \right) \left(c_0 \prod_{i=0}^{n-1} \cos\phi_i \right) \\ &= c_0 \left(m_0 \prod_{i=0}^{n-1} \alpha_i \right) \left(c_0 \prod_{i=0}^{n-1} \eta_i \right) \\ &= c_0 m_{n-1} \alpha_{n-1} c_{n-1} \eta_{n-1} \\ &= c_0 m_n c_n, \end{aligned} \tag{48}$$

which is a sort of Machian form of Eq. (8), and clearly has ample opportunities for further mathematical work (and off-by-one errors).

Practically, however, there are interrelations between the factors in Eq. (48), as they use relative velocities v_i in relation to each parent frame $i - 1$, and also the speed of light c_i, from each parent frame. Also distances r_i are measured in the tangential space dictated by the parent frame in question. It may prove out to be quite difficult to translate all this to more generalized mathematics, as potentially infinite products (when the number of discretization levels is increased) could result in very difficult formulas. I do not have expertise to deal with product formulas so generally, but I am sure that some mathematicians will see some possibilities in here.[29]

[29] As an example, $\cos a \cos b \cos c \cos d = (\cos(a+b+c+d) + \cos(a-b+c+d) + \cos(a-b-c+d) + \cos(a-b-c-d) + \cos(a+b-c+d) + \cos(a+b-c-d) + \cos(a+b+c-d) + \cos(a-b+c-d))/8$.

To make this discussion more concrete, the reduction effect of nested energy frames of Eq. (48) are collected in Table II. The estimated values are highly uncertain, and make more sense in a setting where only a few factors are present (such as studying GPS satellites or similar very accurately known phenomena), and where the velocities and gravitational potentials are more significant.

2. Reduction of radial rest mass at high velocities

As a recap at this point, based on Suntola's analysis, global conservation of energy demands that the radial rest mass $\bar{m}_0$ (or rather its equivalent rest energy, in equilibrium to the global gravitational energy in the local radial direction) of the moving object is actually decreased, balancing the gravitational tangential mass increase, when an object is accelerated in a constant gravitational potential. As implied earlier, I do not yet know for sure whether this reduction in radial rest mass would be rather modeled in terms of diluted wave numbers of Compton wavelengths in the local radial, orthogonal direction, than actual radial "rest mass" reduction (and tangentially the inertial or gravitational mass increase would then be related to de Broglie matter waves), but the mathematical forms should eventually be the same for the model to make sense.

Thus the nested energy frames in the model in Eq. (48) consist of cosine terms, one kind for reduced radial rest mass $\bar{m}_0$, and the other kind for reduced speed of light c, the latter dictated by the tilting of space near mass centers, and parameterized by distances r_i from the great masses M_i. For model use it is nice that the cosine terms are symmetrically even functions, and that they are simply multiplied together starting from the hypothetical homogeneous expanding space $S^3(r_0)$.

As already noted in Eq. (43), the mathematical form of the reduction α_n in the radial rest mass is quite straightforward. With angle φ_n between the total 4-momentum vector p and p_0, the latter of which is the momentum in the local radial direction (projection of the radial expansion momentum, through the chain of frames all the way down to the local frame in question), the energy frame factor α_n decreasing the radial rest mass multiplicatively due to motion, is

$$\alpha_n(v_n) := \cos\varphi_{v_n} = \sqrt{1 - \frac{v_n^2}{c_n^2}}, \tag{49}$$

where, rather cumbersomely at this point, we note the index of the energy frame in question by subindex n, starting from global zero [which is the expanding $S^3(r_0)$ manifold]. Note that it should be clear from context when we are talking about the reduction effect of the frame in question, or the aggregate effect starting from the homogeneous space, so we mix using i and n (and some other

TABLE II To ground the expanding $S^3(r_0)$ model to the physical world, Suntola has tried to estimate the effect of various nested energy frames at present, descending from the perfectly symmetrical cosmos of Eqs. (1) and (23), through the most salient moving mass centers in Eq. (48), to a laboratory on the Earth. The effects are miniscule but nonzero, affirming the challenges ahead in validating the model. See Suntola (2018, p. 245) and the pages preceding it.

	Reduction effect[a] $\alpha_i\eta_i$	Aggregate effect[a] $\prod_{i=0}^{n}\alpha_i\eta_i$
The Milky Way in the extragalactic space[b]	$0.999996 \approx 1-4\times10^{-6}$	0.999996
The solar system in the Milky Way[c]	$0.99999874 \approx 1-1.26\times10^{-6}$	0.99999474
Earth gravitational frame in the solar system[d]	$0.9999999852 \approx 1-1.4808\times10^{-8}$	0.9999947252
Earth geoid in the Earth gravitational frame[e]	$0.999999999303 \approx 1-6.969291\times10^{-10}$	0.999994724500

[a] The number of decimals here are meant to be only illustrative of the weakness of the effects, they are not meant to be taken as significance levels.

[b] Suntola reasons that the "Milky Way is part of the Local Group of galaxies, of which the Andromeda Nebula is the other large member. The Milky Way is estimated to move towards the common center of the Local Group at a speed of about 40 km/s. The Local Group is part of the Local Supercluster, a much larger collection of galaxies including the Virgo Cluster at about 45 million light years from the Milky Way. The Local Group is estimated to move at a speed of about 600 km/s towards the Great Attractor in the direction of the constellation Centaurus. When estimated from the dipole pattern of the microwave background radiation, the solar system is moving at 350–400 km/s relative to the background. The gravitational structure at the extragalactic level is not known in detail. Assuming, that a velocity of 600 km/s has been obtained in free fall from gravitation we end up with factors [as large as these estimates.]". The estimate in the table differs a little from estimate α, using $v = 600$ km/s, $c_0 \approx c$.

[c] Suntola states that the speed of the solar system in the Milky Way is about 220 km/s. Applying the estimated mass of the Milky Way (3×10^{42} kg), the solar mass (2×10^{30} kg), and the distance from the galaxy center (25 000 ly), the factors $\alpha = (1-5.4\times10^{-7})^{1/2}$ and $\eta = 1-9.4\times10^{-6}$, thus $\alpha\eta = 0.99999033$, which differs a little from the value in the table.

[d] Suntola (2018, p. 242) notes that "The motion of the Earth in the solar barycenter frame can be described as Kepler's orbit. Due to the eccentricity of Earth's orbit, the factor $G_{S(B)}$ for Earth in the solar gravitational frame is not a constant but varies during the year, reaching a minimum on January 2 when the Earth is at the perihelion of its orbit. The factor $G_{S(B)}$ in equation (5.7.2:19) gives the correction due to the gravitational state and orbital velocity of the Earth-Moon barycenter frame, which lies within the solar barycenter frame: $G_{S(B)} \approx 1 - 3GM_S/2a_{E(S)}c^2 - (2GM_S/a_{E(S)}c^2)e_{E(S)}\cos\varphi$, where $a_{E(S)}$ is the semimajor axis of the Earth-moon barycenter orbit, and angle φ the true anomaly." M_S is solar mass. Taking only the constant terms one calculates $G_{S(B)} \approx 1-1.481\times10^{-8}$, which is the reduction in the table. Suntola also notes, with detailed references to calculations in the DU book, that "within the Earth frame a difference Δr in the distance to the Sun ($\Delta r \ll r_s$) does not affect the rest energy of mass. Accordingly, there is no daily period in the frequency of clocks on the Earth due to the variation in the distance to the Sun. This is because the effects of gravitation and motion of the locations in the solar gravitational frame cancel each other."

[e] Suntola (2018, pp. 240–241) writes that "The effect of latitude on the gravitational and rotational terms [...] is opposite. Close to the poles, the velocity due to rotation is lower, reducing the rotational velocity term while the gravitational term is increased due to the smaller radius of the Earth. The average value of factor G_E on the Earth geoid (the value applicable to an SI-second standard) is $G_E \approx 1-6.969291\times10^{-10}$."

symbols) for the subindex at will depending on what is most convenient.

Conversely, the tangential mass increase by gaining momentum is calculated by multiplying with the inverse of factor $\alpha_n^{-1} = \gamma_n$, as is usual when using the Lorenz-factors [even though the reasoning leading to Eq. (42) was different, and some would say more direct, than what is commonly offered as explanation]. The v_n and c_n are the values in the energy frame n in question (usually conceptually the parent frame, as from there the reduction of radial rest mass affects the nested frames below).

Suntola maintains that one can indeed handle nested frames like this—e.g., as the solar system has some speed relative to its parent frame, the radial rest masses here are reduced according to the factor in Eq. (49), and so on, until we get to the energy frame we are interested in. The numbers and footnotes in Table II can serve as bringing some physical grounding to the discussion, even if very speculative at this point. If there is no relative speed of the frame, then the effect also vanishes automatically, as $\alpha_n(0) = 1$. Quite straightforward if true, but as mentioned, I have not thought through all the special cases that may happen (such as nested frames at high speeds, or what kind of side effects the discretization granularity apparently may have).

It is important to realize that all this is really calculated and pondered on in various chapters of the DU book. The calculations seem already quite useful for assessing the validity of this otherwise rather theoretical discussion, and for suggesting possible avenues for experimental verification of the claims made.

Of course, the above does not yet explain everything about the elusive concept of mass, as there is still the question of what the rest mass actually consists of, and where it comes from (even though Suntola himself may have very workable ideas here), and the mechanisms of energy frames as some kind of cohesive matter waves are rather fuzzy for me at this point, but nevertheless, there could be some seeds for major clarifications for physics here—provided one could somehow link this to the ongoing discussions about the nature of mass in theoretical physics, and the distinction between this "reduction in radial rest mass" and various observed effects of the increased inertial and gravitational tangential mass, could be clarified further.

3. Reduction of the local speed of light due to deviation from the S^3 geometry

Continuing discussing Eq. (48), the other cosine factor η_n for each energy frame n is a gravitational factor, dependent on the location r_n in space in relation to mass centers. It is found that in this model, the volumetric surface of the $S^3(r_0)$ manifold has to bend in the radial expansion direction near the mass centers (so in effect, the 3-space tilts in the 4D hyperspace toward the mass centers), thus progressively decreasing the local speed of light and radiation.

So evidently the model needs or suggests the gravitational circumstances of wave propagation to modulate the actual limits of the speed of light c, via cosine factors η_n, parameterized by the local aggregate gravitational potential, in a spatially continuous fashion. We have used the symbol η here, as this is a bit reminescent of varying inverse refractive index, even thought the gravitational effect is assumed to be of geometric nature.

This is based on the analysis of energy conservation in free fall, where according to Suntola's analysis, the model is consistent if the 3-space is gravitated/deflected towards the mass center in an angle ϕ_{r_n} (compared to the energy frame "above"), varying according to the space location x in the gravitational potential in question (in the formulas the distance r_n is measured in the straight tangent space of the mass center where the potential originates).

Viscerally, this factor means that the local 3-space near mass centers is actually expanding a bit behind the rest of the space further away from the source of the gravitational potential, and that these dents on a symmetrically perfect $S^3(r_0)$ manifold along the radial expansion direction could actually be real phenomena in the 4-space (or so the mathematics could tell us).

The mathematical form of the proposed gravitational tilting of the local tangential 3-space is quite simple, too. With G as the gravitational constant and M_n as the mass of the source of the gravitational potential, the energy frame factor decreasing the local speed of light, dependent on the local gravitational state, is

$$\eta_n(r_n) := \cos\phi_{r_n} = 1 - \frac{GM_n}{r_n c_n c_0}. \quad (50)$$

When $c_n \approx c_0$, as it most commonly is, we can use Eq. (23) to approximate $c_n c_0 \approx c_0^2 = GM_0/r_0$, and—rather surprisingly—thus relate the gravitational potential generating mass M_n, and the effective mass of the universe M_0, and also the tangential distance r_n to the said mass and the radius r_0 of the $S^3(r_0)$ model, as

$$\eta_n(r_n) \approx 1 - \frac{M_n/M_0}{r_n/r_0} = 1 - \frac{GM_n/r_n}{GM_0/r_0} = 1 - \frac{r_{\pi/2}}{r_n}, \quad (51)$$

where $r_{\pi/2}$ is the critical radius of later Eq. (56). We have typed a few equal forms as they suggest different interpretations: as one's complement to the ratio of relative masses and distances, as one's complement to the relative gravitational potential energy, and as one's complement to the inverse relative distance to the critical radius, respectively.[30]

Note that we have a slight notational inconvenience here, as we are not meaning that in Eqs. (48) and (51) we should equate the apparent mass of the universe M_0 and the great mass M_n at some top frame $n = 0$, as clearly the factor in Eq. (51) would vanish then—but note also that this slight inconvenience in the chosen symbols for the equations suggest that there may be some kind of complementaries present here, too.

For the experts in cosmology and astrophysics, the gravitational factor in Eq. (50) should be familiar as the Schwarzchild radius formula for black holes, but with a factor of 2 and without these different speeds of light. The c_0 again works as an energy conversion factor and is the current expansion speed of $S^3(r_0)$, so the maximum

[30] It is also highly interesting—and reminescent of Dirac's large number hypothesis, see Unzicker (2009)—that Suntola (2018, p. 205) calculates that

$$\frac{\hat{m}}{M_0} = \frac{\hat{r}}{r_0} \quad \left[= H_0 \frac{\sqrt{h_0 G}}{c_0^2} \approx 3.0 \times 10^{-61} \right], \quad (52)$$

where M_0 is the effective mass of the universe and r_0 the radius of the $S^3(r_0)$ sphere, as has been the convention here, and also where

$$\hat{m} := c_0 \sqrt{\frac{h_0}{G}} = \sqrt{\frac{c_0 h}{G}} \approx 5.456 \times 10^{-8}\ \text{kg} = 54.56\ \mu\text{g} \quad (53)$$

$$\hat{r} := \frac{h_0}{\hat{m}} = \frac{\sqrt{h_0 G}}{c_0} = \frac{1}{c_0}\sqrt{\frac{hG}{c_0}} \approx 4.0514 \times 10^{-35}\ \text{m}, \quad (54)$$

are the Planck mass unit and Planck distance unit, respectively, both in DU terms (where intrinsic Planck constant $h_0 := h/c_0 = \hat{m}\hat{r}$ and Hubble parameter $H_0 := c_0/r_0$). Note that the dimensions of the units really match properly here, and there are not many ways to shuffle them around. Compare also to the remarks by Brans and Dicke (1961) quoted in the context of Eq. (22), and also see `https://en.wikipedia.org/wiki/Dirac_large_numbers_hypothesis` and the references therein.

According to WolframAlpha, $\hat{r}$ is of the same order of magnitude than the length of a putative string in M-theory ($\approx 10^{-34}$ m), and here it shrinks proportional to c_0 (and $\hat{m}$ increases proportional to c_0). From an algebraic viewpoint, the relatively large Planck mass $\hat{m}$ (micrograms) may be somehow associated to smaller phenomena, such as electrons, perhaps via some doubling or scaling process (such as 2^{-nd}, 2^{-2^n}, or its variations), and to larger objects, such as neutron stars, in an inverse process (2^{2^n}, etc.), as studied by Ari Lehto, among others. This is of course highly speculative, but frankly, somebody has to explore these possibilities, too.

speed of light in the cosmos at large, in principle available everywhere, and c_n is the speed of light on the energy frame being analyzed (getting its value by descending in layers from "the cosmos above", so to speak). Most often $c \approx c_0$, as the gravitational singularities are so strong that the nested frames have not had much of an effect at those distances.

Recall that this $\cos\phi_{r_n}$ term in Eq. (50) is just a factor along the chain when multiplying the c_0 down from the perfectly symmetrical $S^3(r_0)$ at the cosmos level, to the local speed of light c, as described in the factorized structure of the rest energy in Eq. (48).

The magnitude of c in the radial direction to the local 3-space is thus the local projection of the radial hyperspeed c_0, suggesting interpreting the gamma matrices and the relevant Dirac algebra in terms of this deflected axis [so Suntola's "imaginary direction" may be the first gamma matrix γ_0, in some settings, such as the geometric approach pursued by Anthony Lasenby in unifying the fundamental forces, see (Doran and Lasenby, 2003) and their other publications and presentations].

Based on his analysis, Suntola states that as $c_n \cos\phi_{r_n}$ is the reduced speed of light at distance r_n from M_n, importantly then $c_n \sin\phi_{r_n} = c_n(1-\cos^2\phi_{r_n})^{1/2}$ is the achieved velocity on free fall at r_n, which is also the same as escape velocity from that distance.[31]

Suntola makes a very intriguing suggestion here: in the free fall, where a test mass is falling towards the mass center in a 4D-potential well, the velocity can supposedly increase without bounds, or at least catch the local speed of light, which decreases as the gravitational tilting of the 3-space gets more severe. Apparently there is a special threshold point in Eq. (50) if the deflected angle attains the value $\phi_{r_n} = \pi/4 = 45°$ [where $\cos(\pi/4) = 1/\sqrt{2}$], as there the velocity of free fall could, at least mathematically in this model, catch the local speed of light c and "things would happen".

This special threshold radius is clearly

$$r_{\pi/4} := \sqrt{2}\,(1+\sqrt{2})\frac{GM_n}{c_0^2}, \tag{55}$$

which Suntola notes that curiously, neutron stars may have. Calculating using the current estimated variability of neutron star mass $M_n \approx 2.187$–4.176 ($\times 10^{30}$ kg), this threshold radius $r_{\pi/4} \approx 5\,546$–$10\,590$ m, which is rather close to the current estimations of neutron star radius of 11–13 km (which are quite uncertain themselves as the equation of state of neutron stars is not well known).[32]

Even deeper in the 4D potential well, there is a special critical radius in the same Eq. (50) at asymptotic aggregate angle $\phi_{r_n} = \pi/2 = 90°$, where the local tilted 3-space would be parallel to the expanding 4-radius (with regards to one of the dimensions, not all at the same time, of course) and thus the local speed of light would asymptotically slow down to zero. This special critical radius is clearly half the Schwarzchild radius of black holes by construction, as

$$r_{\pi/2} := \frac{GM_n}{c_0^2} = \frac{M_n}{M_0} r_0, \tag{56}$$

where we have again used Eq. (23) to approximate $c_n c_0 \approx c_0^2 = GM_0/r_0$, similarly to Eq. (51). Compare this also to Eq. (52), suggesting that there could be deep similarities between the micro and macro scales. Perhaps the "cutoff" needed in the highly accurate quantum mechanics calculations for the properties of pointlike micro structures such as the electron to not result in infinities, could have some physical basis like this (this is highly speculative at this point, however).

It is not known what happens inside the critical radius $r_{\pi/2}$ in this model. For such extremely dense phenomena as black holes, where the space could bend in an angle $\pi/2$ to be parallel to the expansion direction at the critical radius—where the local speed of light would come to a standstill—is there some kind of easing when approaching the center of the massively dense 3-ball under consideration, or is there some kind of inversion of space, or a possibility for very deep wells, perhaps even tunnels or wormholes to other parts of the $S^3(r_0)$ sphere, that I do not know. In any case the hypothetical conditions there are so severe that it is difficult to imagine what would happen to the various known forces of nature at the smallest length scales, for example.

For the space near the critical radius, Suntola calculates that under this model, there may be slow orbits which maintain the mass of the black hole, which is a very interesting proposition. The "event horizon" gets quite a different interpretation here than has been customarily assumed, as now the local speed of light itself can vary, approaching zero.

Fortunately, for quite ordinary settings, such as doing celestial mechanics or studying GPS satellites, the gravitational potentials and their gradients are not very large. The deflections of the local 3-space towards the mass centers and thus the implications for the speed of light, are

[31] There is an intriguing symmetry here, as if the velocity obtained from free fall is $v = c_0 \sin\phi$, then $\cos\phi = (1-\sin^2\phi)^{1/2} = (1-v^2/c_0^2)^{1/2}$ and thus $\eta = \alpha$ in Eq. (48), so we have to be careful which factor to use—Suntola infers that actually the radial rest mass (and thus also symmetrically the tangential relativistic mass) is not affected in the free fall, only the local speed of light is, resulting in the increase in velocity, which is quite though-provoking suggestion in relation to the research on black holes, for example.

[32] Note that $\sqrt{2}(1+\sqrt{2}) = \sqrt{2}+2 \approx 3.414$, where $\delta_S := 1+\sqrt{2}$ is the so called silver ratio.

ordinarily rather small.[33]

As we know the tilting angle ϕ_{r_n} of the space as a function of distance r_n from the mass center M_n, we can calculate the dent in the $S^3(r_0)$ manifold quite accurately. For simplicity, we first assume that we are in the homogeneous "uppermost" level and calculate the derivative of radial displacement ζ_n (zeta) with respect to the distance r_n, as the tangent of the deviation angle ϕ_{r_n} in Eq. (50),

$$\frac{d\zeta_n}{dr_n} = \tan\left[\arccos\left(1 - \frac{GM_n}{r_n c_n c_0}\right)\right] = \frac{\sqrt{\frac{r_n}{r_{\pi/2}/2} - 1}}{\frac{r_n}{r_{\pi/2}} - 1}, \quad (57)$$

where we have utilized the critical radius $r_{\pi/2} = \frac{GM_n}{c_n c_0}$ to make it more concise.

The derivative $d\zeta_n/dr_n$ of radial direction correctly vanishes when $r_n \to \infty$, and grows without bound (describing a deep well, note the positive direction of the distance) when approaching critical radius $r_n \to r_{\pi/2+}$ from the right.[34]

Note that the critical radius is typically rather small, as $r_{\pi/2}(\text{Earth}) \approx 4.44$ mm ($M_{\text{Earth}} = 5.97 \times 10^{24}$ kg), $r_{\pi/2}(\text{Sun}) \approx 1\,477$ m ($M_{\text{Sun}} = 1.988 \times 10^{30}$ kg), and $r_{\pi/2}(\text{electron}) \approx 6.765 \times 10^{-58}$ m ($M_{\text{electron}} = 9.109 \times 10^{-31}$ kg). Naturally, the critical radius calculation only applies when we are outside of the mass surface of the object, not under the surface. This is because when the gravitational potential originates from inside a star or a planet, for example, and we are considering some location underground, only the deeper core of the planet (as ordinary 3-ball) contributes to the gravitational potential, as the contribution of the outermost shell (compared to the underground location under consideration) cancels away, as usual when integrating the Newtonian $1/r$ potential in an Euclidean 3-space. So nearer to the center of the planet the potential weakens, smoothing the gravitational potential well to a concave cup. But outside the planet or star surface, the critical radius of the total mass is a good measure to relate the distances to, even if it is quite tiny in comparison.

To see the form of the resulting dented surface, we can integrate Eq. (57),

$$\int \frac{d\zeta_n}{dr_n}\, dr_n = 2\, r_{\pi/2}\left(\sqrt{\frac{r_n}{r_{\pi/2}/2} - 1} - \operatorname{artanh}\sqrt{\frac{r_n}{r_{\pi/2}/2} - 1}\right) + C \quad (58)$$

which has two special points, a root $r_n = r_{\pi/2}/2$ and a pole at the critical radius $r_n = r_{\pi/2}$. The inverse hyperbolic tangent (artanh) is a multivalued function and hence requires a branch cut in the complex plane, but choosing the principal value for the branch $r_n > r_{\pi/2}$ we can integrate between any two points outside the critical radius, and it will actually result in a real value for the displacement ζ_n (as it should).[35] Note that even though the derivative in the Eq. (57) eventually vanishes when $r_n \to \infty$, its integral in Eq. (58), and thus the negative displacement, actually grows without bound (but very slowly) in the same process, presumably due to Eq. (50) not needing to consider very large distances where the spherical curvature of the $S^3(r_0)$ manifold would start having an effect. We should perhaps use the spatial distance from Table I here, to have the current 4-radius r_0 have an effect at very large distances.

We can choose some reference distance, for example $R\, r_{\pi/2}$ relative to the critical radius $r_{\pi/2}$, and calculate the relative deviation $\Delta\zeta_r$ in the radial direction. Integrating Eq. (57) between a closed interval,

$$\begin{aligned}\Delta\zeta_r &:= \int_{R r_{\pi/2}}^{r_n} \frac{d\zeta_n}{dr'_n}\, dr'_n \\ &= 2\, r_{\pi/2}\left(\sqrt{\frac{r_n}{r_{\pi/2}/2} - 1} - \sqrt{2R-1}\right. \\ &\qquad \left. - \operatorname{artanh}\frac{\sqrt{\frac{r_n}{r_{\pi/2}/2} - 1} - \sqrt{2R-1}}{1 - \sqrt{\frac{r_n}{r_{\pi/2}/2} - 1}\sqrt{2R-1}}\right), \end{aligned} \quad (60)$$

where we have used the identity

$$\operatorname{artanh} u \pm \operatorname{artanh} v = \operatorname{artanh}\frac{u \pm v}{1 \pm uv}, \quad (61)$$

to collect the terms.[36]

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[33] Suntola (2018, p. 139) argues, using Eq. (50), that at the surface of the Earth, the velocity of light is reduced by about 20 cm/s compared to the velocity of light at the distance of the Moon from the Earth. The velocity of light at the Earth's distance from the Sun is about 3 m/s lower than the velocity of light far from the Sun. The frequencies of atomic clocks are assumed to vary in the same proportions, masking the difference, in this model. The daily and yearly variations are calculated and pondered on in the DU book.

[34] The mathematical form of Eq. (57) suggests that approaching the critical radius $r_n \to r_{\pi/2-}$ from the inside (from left), the derivative decreases without bound, and curiously there is a zero inside the critical radius at $r_n = r_{\pi/2}/2$. For distances $r_n < r_{\pi/2}/2$, the derivative will be purely negative imaginary, crossing $r_n = 0$ at $-i$ and vanishing to zero when $r_n \to -\infty$. However, as this derivative Eq. (57) is based on Eq. (50), it may not be valid inside the critical radius (let alone $r_n < 0$).

[35] The neat minimalism of the power series expansion

$$\begin{aligned}\operatorname{artanh}(x) :&= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}, \quad |x| < 1 \\ &= \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(1-x) \end{aligned} \quad (59)$$

together with the similarly neat power series expansion of the exponential function, suggests (to me anyways) that there could be some iterative scale-free processes involved in here.

[36] One could simplify it further by a change of coordinates $r_n =$

The effect of the reference distance R on $\Delta\zeta_r$ diminishes as $\propto R^{-2}$, so even though the integral can grow without bounds, practically there should be a point where one could rely also on the absolute values to be representative of the actual dent of $S^3(r_0)$ also closer to the critical radius (if the model is correct in the first place, that is). Suntola himself uses a valid approximation of Eq. (60) when both $r_n \gg r_{\pi/2}$ and $R_n \gg r_{\pi/2}$, as

$$\tilde{\Delta}\zeta_r = \int_{R_n}^{r_n} \frac{d\zeta_n}{dr'_n}\, dr'_n \approx 2\sqrt{2}\, r_{\pi/2}\left(\sqrt{\frac{r_n}{r_{\pi/2}}} - \sqrt{\frac{R_n}{r_{\pi/2}}}\right) = 2\sqrt{2r_{\pi/2}}\left(\sqrt{r_n} - \sqrt{R_n}\right), \quad (64)$$

for the surface shape [Eq. (4.1.9:4) in DU book].

Concretely, using either Eq. (60) or (64), one can then calculate that under this model, the Sun is lagging Earth about 39 000 km behind, in the expanding radial direction of the $S^3(r_0)$ geometry.[37] More visually, this difference in the radial direction is about three diameters of Earth, compared to the ordinary distance of over 11 700 diameters of Earth in the tangential space between Earth and the Sun.

Even though I could already replicate some of the model predictions in the above, clearly I do not yet follow all the reasonings and their implications in terms of the nested energy frames. Especially the orbital velocities I have not analyzed here. Also the various visual diagrams, while very illustrative, may mask some algebraic problems or circular arguments, and while I have no particular reason to suspect that to be the case, I would like to have a mathematical certitude in these concepts. For example, I have not thought through how the gravitational potentials are generated in moving frames, and how are they responded to in moving frames. There is a lot of ground to be uncovered while studying the work further.

As Suntola's system-oriented approach to gravity does not posit some fixed qualitative thresholds between the macro and the micro—even the nested energy frames should be mostly epistemic, not strictly ontic—but insists on the global being linked to the local via the expanding geometry of space, there may be all kinds of examples to be discovered in the DU book where even the most common concepts in physics could be seen from a new perspective.

So with that in mind, I will now finally note how the c_0 is again key to many other parts of the model, such as how time is measured.

IV. THE PHYSICS OF DIVERGING TIMES

There is consensus that observations have verified both kinematic and gravitational time dilation.[38] Both are important effects that need to be accounted for.

One of the major points of DU is that according to the model, the physical realness of time dilation is being exposed as clocks simply losing time depending on the kinematic and gravitational state.

As an observed phenomenon, this is not a very controversial take. Even Bohm (1996 [1965], p. 127) states that "all physical, chemical, nervous, psychological, etc., processes will be subject to the same Lorentz transformation that applies to clocks." However, in the literature there seems to be a sort of confusion between what is physical and what is mathematical, as the Lorentz transformation is often times taken as a physical effect, as if the time itself would be warped somehow (with quite weak suggestions for possible causation mechanisms, really). This results in passages such as in Bohm (1996 [1965], p. 131) (taking into account both special and general relativity):

> If the rocket observer were watching the fixed observer, he would then see the life of the latter slowed down at first and later speeded up, but he would find over the whole course of the journey that the effect of the speeding up more than balanced that of the slowing down.

$r'_n\, r_{\pi/2}/2$ relative to half of the critical radius, resulting in

$$\Delta\zeta'_n = \sqrt{r'_n - 1} - \sqrt{R' - 1} - \operatorname{artanh}\frac{\sqrt{r'_n - 1} - \sqrt{R' - 1}}{1 - \sqrt{r'_n - 1}\sqrt{R' - 1}}, \quad (62)$$

which suggests that the form

$$\begin{aligned}\Delta\zeta''_n &= x - y - (\operatorname{artanh} x - \operatorname{artanh} y) \quad (63)\\ &= x - y + [\ln(1-x) - \ln(1+x) - \ln(1-y) + \ln(1+y)]/2\\ &= x - y + \frac{1}{2}\left(\int_1^{1-x}\frac{dx'}{x'} - \int_1^{1+x}\frac{dx'}{x'} - \int_1^{1-y}\frac{dy'}{y'} + \int_1^{1+y}\frac{dy'}{y'}\right)\end{aligned}$$

could become important in the future. There are also special distances such as $r_n = 5\, r_{\pi/2}/2$, resulting in simpler forms, or $r_n \approx 1.2196\, r_{\pi/2}$, where the real part of the open integral in Eq. (58) vanishes, but which may be not more than mathematical curiosities at this point.

[37] Suntola (2018, p. 138) calculated that 'the Sun dips about 26 000 km further into the fourth dimension than does the Earth, which is about 150 000 km "deeper" than the [dwarf] planet Pluto', which is in the same order of magnitude, but less than the numbers I got. Here I have used the critical radius of the Sun, $r_{\pi/2} \approx 1\,477\,\mathrm{m}$ [calculated earlier using the mass of the Sun and Eq. (56)], and integrated the resulting radial deviation $\Delta\zeta_n$ using Eq. (60) or (64), between the mean radius of the Sun, at $R_n = 6.957 \times 10^8\,\mathrm{m}$, and mean Earth-Sun distance at $r_n = 1.488 \times 10^{11}\,\mathrm{m}$. I am not taking into account the nested smaller dent that Earth's own gravitational field produces, and one could also take into account that the center of the Sun is lower in the radial direction than its surface. Also I do not yet know how relevant for the visual picture is the slight asymmetric position of the total gravitational field of the solar system around the center of the total mass of the solar system (its barycenter).

[38] For a nice summary of time dilation experiments done, see https://en.wikipedia.org/wiki/Hafele-Keating_experiment and the discussions in the DU book.

> He would not therefore be surprised to discover on meeting with his twin [the fixed observer] that the latter had experienced more of life than he had.

Suntola reasons that in an expanding $S^3(r_0)$ model, there actually is a totality of space and matter *at this very moment*, and these kind of thought experiments would need to take into account that whole to be able to reason about the parts. For the twin paradox (that is not a paradox according to Bohm, but can be solved through rather convoluted arguments—in my view—relating the velocities, accelerations and different kinds of apparent gravitational fields), it would be easier to simply think about the matters at hand separately in relation to the whole, such as the observed distances (recall Table I), kinematic states (as the rocket clearly is in a different kinematic state in relation to the expanding $S^3(r_0)$ geometry, in this model), and gravitational states (where the fixed observer on Earth would stay in a rather stable state). According to the model studied here, the rocket observer would simply age less due to physical processes running more slowly at speed, on both ways, and the effects of different gravitational potentials are quite weak (and the rocket having some near encounters with the gravitational fields of great masses in space would just amplify the difference).

The impetus here is to take the measurement instruments and perceptions seriously as physical entities and natural processes, and propose at least some preliminary mechanisms that could be due to gravity (as opposed to merely mathematical, phenomenological regularities) and thus aiming to clarify the important concepts of time, matter, and space further.[39]

A. Kinematic and gravitational time dilation

Inferring common parlance from Wikipedia, kinematic time dilation refers to the observation that "the rate of a clock is greatest according to an observer who is at rest with respect to the clock. In a frame of reference in which the clock is not at rest, the clock runs more slowly, as expressed by the Lorentz factor."

Suntola proposes that the kinematic effect is due to clock's radial rest mass decreasing multiplicatively as tangential kinetic energy is increased (in a constant gravitational potential), to counter the suggested decrease in global gravitational energy due to increased tangential motion in $S^3(r_0)$, as described earlier. The connection to the varying clock frequencies is via the postulated $E_0 = c_0 m_0 c$ structure of rest energy, where m_0 (and $p_0 = m_0 c$) is a factor. This is analyzed in a bit more detail in a moment.

By contrast, again referring to common understanding displayed in Wikipedia, gravitational time dilation refers to the observation that "an increase in gravitational potential due to altitude speeds the clocks up. That is, clocks at higher altitude tick faster than clocks on Earth's surface." To get things in order, it is important to realize that potentials are often modeled as $-1/r$, so stating an increase in the potential with the altitude, means the same as the potential getting weaker away from the mass centers, as it should.

Suntola suggests that actually the speed of light is higher with the altitude, as also discussed before, and that clocks tick faster in direct proportion to the increase in local speed of light (this is the seminal point), hiding the variation from local observers. The difference in elapsed time is found out in comparison with clocks in other kinematic and gravitational states, exactly as is observed. This is again a very straightforward suggestion—provided one can accept, if not yet approve, that coupling gravitational states, varying speed of light, and clock frequencies together could result in coherent thinking down the road.

So in this model, the gravitational effect is due to gravitational potential getting stronger (when traveling across the gravitational potential towards mass centers), tilting

[39] Compare to Bohm (1996 [1965], pp. 132–133):

> "We see then that there is actually nothing paradoxical in the relativistic conclusion that an accelerated clock will register less time in passing between two points than would an unaccelerated clock passing between the same points. This is possible because in relativity theory time is not an absolute, with a universal moment "now," the same for all coexistent observers. Rather it is a much more subtle sort of notion, which can be different in relation to different frames of reference. There is room for many different kinds of time, as registered by clocks and physical processes that are subject to different kinds of movement. In many ways physical time thus begins to show some of the properties of our own experience with time in immediate perception. Thus, it is well known that a given interval as measured physically by a clock may seem long or short, an eternity or a mere moment, depending on how much is happening during the interval. (In Section A-3 of the Appendix we shall go into this problem of perception of time in more detail.) Until the development of the theory of relativity, it seemed that physical or chronological time did not share such a relativity and dependence on conditions. But now we see that this is because it had been studied only in the limited domain of low velocities. As the domain is broadened to include velocities appreciable in comparison with c, we have begun to find in chronological time a dependence on conditions that is not entirely dissimilar to what we experience in immediate perception. In other words, all forms of time, including the chronological and the perceptual, are means of ordering actual events and measuring their relative duration. The notion that there is one unique universal order and measure of time is only a habit of thought built up in the limited domain of Newtonian mechanics. It is valid in that domain, but becomes inadequate as the domain is extended. And perhaps as the domain is broadened still further we may well have to modify our conceptions of time (and space) yet more to enrich them, and perhaps to change them radically, in such a way that even current relativistic notions are treated as approximations and special limiting cases."

the space locally away from globally perfect $S^3(r_0)$ geometry, affecting the local speed of light, which in turn affects atomic clocks and other measurement instruments (or most of physics!) via the postulated $E_0 = c_0 m_0 c$ energy conversion relations, where c is a factor. This important point is briefly elaborated in the following, as it may reveal some promising theoretical degrees of freedom in fundamental physics.

As the energy conversion relation is bilinear in m_0 and c, the effects of all the nested energy frames above the analyzed frame are simply aggregated together by multiplying all the cosine factors together, as in Eq. (48). This could be a deceivingly simple operation, as calculating some of the terms involve using values from the parent frames.

It is important to emphasize that the suggested relation of clocks to their kinetic and gravitational states is indeed via the radial rest energy $c_0 p_0$, not the total energy where effects due to tangential motion are still ordinarily included. Kinetic energy increases in motion by definition, and it has of course effects such as increasing inertia and gravity in synchrony, as the inferred tangential mass is then $m_\perp = p_\perp / v$ (or its energy equivalent), where $p_\perp$ is the observed 3-momentum (that increases when one approaches very high velocities). I am not perfectly happy about the terminology such as radial rest energy here, but am quite pleased that the ordinary tangential 3-space and its orthogonal complement direction in 4-space are at least starting to get some shape for theoretical analysis. In this model, the effects of rest mass are indeed in the local radial direction, thus in relation to the global gravitation and the rest of space via the $S^3(r_0)$ geometry.

In addition, the model also predicts a global decrease in the speed of light c_0 slowly over time, as has become apparent in Eq. (29) and the analysis carried out in its vicinity, which also affects the clocks in the long term, such as in geological time scales.

B. Atomic clocks dependent on motion and gravitation

Analyzing the behavior of atomic clocks is a very important step, to be able to start reasoning in the aforementioned situations. First of all, Suntola correctly notes that one can model the frequency f of an atomic clock as proportional to the difference dE in energy between two hyperfine levels of the atom,

$$f = \frac{dE}{h}, \tag{65}$$

where h is the Planck constant. He then suggests that the energies involved are proportional to the rest energy equivalents (of oscillating outer shell electrons, or some equivalent masses that affect the hyperfine structure determining the state transitions)[40],

$$dE \propto c_0 m_0 c, \tag{66}$$

and observes that at first it does not bode well for the model. The reasoning to see the conflict, and the insight leading to the rather surprising idea to resolve the problem, goes as follows.

Combining the above relations, the tick frequency of an atomic clock,

$$f \propto \frac{c_0 m_0 c}{h}, \tag{67}$$

is evidently correctly linearly proportional to the rest masses m_0 involved in the oscillators. The hypothesized decrease in the rest energies and masses causes then the dilated measurements of time in accelerated clocks, reproducing the Lorenz-transformed phenomenology accurately. The frequency is also correctly proportional to the local speed of light c, enabling the speed of light observations to be constant in a changing gravitational potential as we have become accustomed to, as both the measure and the measured then vary together.

But from a cosmic perspective, the radial $S^3(r_0)$ expansion velocity c_0 changes, too, and thus the frequencies of atomic clocks are eventually proportional to the square of the speed of light c_0^2, thus being far from linear as needed for the deduced model of the atomic oscillators to match with the observations universally, in this cosmological model. There is a conflict.

However, with great insight, from a separate analysis solving the Maxwell's equations in relation to a novel 4D model of the electron (or so I gather at this point), Suntola infers that actually the model can resurrect known physics by simply deriving c_0 being a factor in the Planck constant h. This is a very productive proposition, as then one can immediately see that the clock frequencies could really be directly proportional to $m_0 c$, as then, combining all of the above,

$$f = \frac{dE}{h} \propto \frac{c_0 m_0 c}{h_0 c_0} = \frac{m_0 c}{h_0}, \tag{68}$$

where h_0 is an intrinsic Planck constant (a DU concept), from where the evolving energy conversion factor c_0 has been removed. Therefore, the atomic clocks could actually vary according to the local speed of light and the kinetic state, thus reproducing the relativistic observations within the current measurement errors. The DU

[40] See, for example, https://en.wikipedia.org/wiki/Hyperfine_structure#Use_in_defining_the_SI_second_and_meter. According to the current SI standard, one second is defined to be exactly 9 192 631 770 cycles of the hyperfine structure transition frequency of cesium-133 atoms, and the meter is defined as the length of the path travelled by light in a vacuum during a time interval of 1/299 792 458 of a second.

book has dedicated many pages to calculating the various implications of this prospect, and as astonishing as it may sound, from the looks of it, it seems plausible that this may turn out to be the correct idea.[41]

Clearly this is not some random suggestion—one can see the proposed intrinsic structure of the Planck constant conceivably being an essential element in simplifying a lot of the fundamental equations in physics further.

C. Speed of light as a factor in Planck constant

As an example, the Klein-Gordon equation, which the components of all free quantum fields obey in the (relativistic) quantum field theory, is (in position space),

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right)\psi(t,x) = 0, \tag{69}$$

and now it could perhaps attain a more fundamental form

$$\left(\frac{\partial^2}{\partial x_0^2} - \nabla^2 + k_0^2\right)\psi(x_0,x) = 0, \tag{70}$$

exactly, without resorting to using natural or geometric units (where several constants are conventionally set to unity).

Here I have simply used the definitions $x_0 := ct$ (which is the advance in the local radial direction up to time t), $h := h_0c_0$ (the proposed structure of the Planck constant), and the wave number $k_0 := 2\pi/\lambda_0 = m_0c_0/\hbar = m_0/\hbar_0$, where $\lambda_0 := h/(m_0c_0) = h_0/m_0$, so the Compton wavelength equivalent of radial rest mass m_0. I am not sure about the c vs. c_0 (we may need to introduce a proportionality factor χ here), but as applying quantum field theory and the Dirac equation properly adds all kinds of relativistic complications here in any case, that I am not at all familiar with, I just note that simply on a surface level, the mathematical simplicity here is striking. It is not unreasonable to ask whether this gives some new physical insight into the nature of the related wave operator (also called d'Alembertian), as if operating in an expanding $S^3(r_0)$ geometry in the local radial direction. The 3-space and the radial direction may have opposite signs in these kinds of relativistic operators simply due to everybody riding along with the expansion.[42]

As a second example, using the proposed intrinsic Planck constant h_0 and the identities in the DU book, the fine structure constant α then attains the dimensionless form,

$$\alpha := \frac{e^2\mu_0}{2h_0} \approx \frac{1}{4\pi^3\,1.1049} \approx \frac{1}{137.0360}, \tag{71}$$

where the last approximations are due to Suntola's novel "4D antenna solution" still having a geometric factor of 1.1049, that is at the present moment yet without a structural explanation (other than it is close to unity, so compatible with interpreting it as a hypothetical geometrical factor of a conceptual isotropic antenna).[43]

Also the Bohr radius of a hydrogen atom is then

$$a_0 := \frac{h_0^2}{\pi\mu_0 e^2 m_e} \approx 5.291\,772\,109\,03(80) \times 10^{-11}\ \text{m}, \tag{72}$$

where the proposed structure of electric permittivity ε_0 in Eq. (21) has been used. Note that as DU suggests that the radial rest mass m_e changes according to the energy frame in motion, Eq. (72) predicts an increase in dimensions when an object is accelerated to high tangential velocities—not unlike the Lorentz-FitzGerald length contraction, but in opposite direction and applying to all dimensions equally.[44] Also as Eq. (72) is independent of c_0 here, this seems to suggest that atoms do not expand with the space.

Overall there are a lot of highly interesting mathematical and philosophical developments in the DU book,

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[41] Compare to Bohm (1996 [1965], p. 139): "if the events are inside each other's light cones, [...] all observers will agree on which is earlier and which is later (so that there can be no ambiguity in the order of causally connected sets of events)." and further on (p. 36), "Within a certain [aforementioned] limited domain we have found that it is indeed possible to ascribe a single, universal, well-defined time order to events, as is implicit in the notions described above. Within this domain, many different observers, using different instruments and procedures, all agree within an appropriate experimental error which events are copresent, which are before others, and which are after. In other words, there is a good factual basis for the assumption of the *chronological order* of a unique past, present, and future, the same for all events of every kind, regardless of where they take place and how they are observed." (emphasis in the original)

[42] Lately I have been studying the structure of gamma matrices and became increasingly curious about these relationships, as the gamma matrices (in Dirac representation) have forms $\gamma_0 = \mathbf{e}_0$, $\gamma_1 = \mathbf{e}_{10}$, $\gamma_2 = \mathbf{e}_{20}$, and $\gamma_3 = \mathbf{e}_{30}$, so $i\gamma_5 = -\mathbf{e}_{123}$ and $C = i\mathbf{e}_2$, and in some other presentations the role of γ_0 and γ_5 are reversed. Together, the gamma matrices and their product γ_5 represent a five-dimensional space, and there are obvious geometric interpretations for points, planes and expanding volumes.

[43] Specifically, Suntola proposes that Planck constant $h = 1.104\,905\,31\ 2\pi^3 e^2\mu_0 c = h_0 c = 6.626\,070\,15 \times 10^{-34}$ J/Hz [kg m^2/s], where speed of light in vacuum $c = 299\,792\,458$ m/s, elementary charge $e = 1.602\,176\,634\times10^{-19}$ C, and vacuum magnetic permeability $\mu_0 = 1.256\,637\,062\,12(19) \times 10^{-6}$ N/A^2. This results in intrinsic Planck constant $h_0 \approx 1.1051\times2\times10^{-42}$ kg m. Note that I am not yet certain of the c_0 vs c, as the Planck constant is measured (and nowadays simply defined) in a certain way where the relation to a possibly changing speed of light can be quite obscure.

[44] I note, again, that I have not yet reviewed the relevant tests of relativity here. I am painfully aware and emphatic, however, what kind of complications all this would mean, if true, for the ongoing efforts in defining the SI units ever more accurately, for example. For discussions on length contraction, see also Brown and Pooley (2004).

ranging from very specific proposals to the most general ideas, but at this point it is practical to try to cover only these few here.[45]

V. DISCUSSION AND FURTHER WORK

In light of what has been said above, it is quite understandable that these kind of analyses are usually completely missed from even very creative and novel theories of gravity. It has been convenient to use natural or geometric units (where constants such as c and h have been set to unity) in a lot of theoretical work, thus missing opportunities to see these kind of mathematical possibilities of variance across the levels.

Apparently there is now a non-zero possibility that the four-dimensional spacetime continuum model, which has proven out to be so very useful and accurate during the last century, should be extended with a five-dimensional model, where the first, radially expanding dimension could prove out to be actually the primary one, and the timelike dimension could be delegated to its proper place as a conceptual, modeling dimension, rather than necessarily reifying it with some fundamentally physical, deterministic and at times reversible existence as is often done.

Also the contemporary studies into the nature of the expanding universe (the circumstances around the Big Bang and whether the expansion accelerates or not, and what may drive the expansion) may find some fresh ideas in these rather serious attempts at explanation described above.

If the speed of light actually varies depending on the gravitational state, and if clock frequencies (and most of physics, who knows at this point) are simply varying in proportion in the energy frames of the observer and the observed [as studied from a common rest frame, as is the convention already in using Earth-centered inertial (ECI) coordinate frames when studying GPS clocks], and especially when objects are accelerated to high velocities, things might get very interesting for the physics and cosmology.[46] In any case, I am fairly certain that this is an interesting prospect and can be studied mathematically in a principled theory work. As Suntola has proceeded to compare DU with all kinds of recorded observations he has been able to find during his multi-decade work, the model is in quite a different experimental position than many other fashionable theories proposing some exotic physics.

One of the most foreign suggestions here seems to be the distinction between radial and tangential mass, but which too may follow—quite naturally, even—from accommodating the various features of mass to the available degrees of freedom in the expanding $S^3(r_0)$ geometry, as Suntola has done. Studying the validity of this suggestion and enumerating its possible consequences should be among the major thrusts in future work.

We can delegate some of this discussion to Bohm (1996 [1965], p. 140–141), who ends his book on special relativity with the following three paragraphs, quoted here in full:

"In all maps (conceptual or otherwise) there arises the need for the user to locate and orient himself by see-

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[45] For example, orbital velocities and various dynamics of the solar system have not been studied here. Also the interpretations of electromagnetism and double-slit experiments have been skipped. Analyzing the blackbody radiation, Suntola proposes that the Boltzmann constant k_B also has the speed of light as a factor, but now squared: $k_B = c_0 k_{B_0} c$. This also seems a very productive proposition, as many formulas seem to simplify, similarly to introducing the intrinsic Planck constant h_0.

As an example, the Hawking radiation temperature is then

$$T_H = \frac{\hbar c^3}{8\pi G M k_B} = \frac{c_0 \hbar_0 c^3}{8\pi G M c_0 k_{B_0} c} = \frac{\hbar_0 c^2}{8\pi G M k_{B_0}}, \qquad (73)$$

where the familiar GM/c^2 form is evident.

[46] Let's appreciate Bohm's thinking here (Bohm, 1996 [1965], p. 140) (emphasis in the original):

"In a similar way the physicist's notions of space and time are based on a reconstruction, in accord with appropriate geometrical, dynamical, and structural principles that have been abstracted from a wide range of past experiences. These too have errors that have to be corrected on the basis of further observations, and can be subjected to fundamental structural alterations, as experience is extended into new domains. And, likewise, the map is never complete. Indeed, it is based only on what is past and gone. But when all of this is put into the structure of a good conceptual map, it can serve as a general guide for what to expect in the future. However, to see what the future is really like, we must, of course, wait until it actually takes place. And from time to time, there will be surprises, not corresponding at all to what is on our map.

The difference between a map and the region of which it is the map is so self-evident that no one is likely to confuse the map with what it is supposed to represent (any more than someone is likely to confuse a picture of a meal with a real meal that can nourish him). But our ideas of space and time (whether gained in common experience or in physical research) seem to be comparatively easily confused with what actually happens. Thus, when Newton proposed the idea of absolute space and time physicists did not say that this is only a kind of conceptual map, which may have a structure that is partly true to that of real physical processes and partly false. Rather, they felt that *what is* is absolute space and time. Now that this notion has been seen to have only a limited degree of validity, the tendency is probably to feel that *what is* is relativistic space-time, as shown in the Minkowski diagram.

Much confusion can be avoided on this point if we say that both Newtonian and Einsteinian space-time are conceptual maps, each having a structure that is, in its domain, similar to that of real sets of events and processes that can actually be observed. Room is then left in our minds to entertain the notion that as physics enters new domains, still other kinds of conceptual maps may be needed."

ing which point on the map represents his position and which line represents the direction in which he is looking. In doing this, one recognizes, in effect, that every point and direction of observation yields a unique perspective on the world. But with the aid of a good map having a proper structure, one can relate what is seen from one perspective to what is seen from another, in this way abstracting out what is invariant under change of perspective, and leading to an ever-improving knowledge and understanding of the actual character of the territory under investigation. Thus, when two observers with different points of view communicate what they see, they need not argue, offering opinions as to which view is "right" and which view is "wrong." Rather, they consult their maps, and try to come to a common understanding of why each man looking at the same territory has a different perspective and comes therefore to his own view, related in a certain way to that of the other. (Of course, if after reasonable efforts they cannot do this they may begin to suspect that they may need maps with different structures.)

In Newtonian mechanics the importance of the location and perspective of the observer was very much underemphasized. Of course, physicists have probably always realized that each observer does actually have a perspective. However, they may have felt that such a perspective need play no part in the fundamental laws of physics. Rather, they assumed that a physical process takes place in an "absolute" space and time that is independent of the way in which it is measured and observed, so that the perspective of the observer (or of his instruments) does not appear at all in these laws. On the other hand, in Einstein's point of view, it is clear that any particular example of a Minkowski diagram is a map corresponding to what will be observed in a system moving in a certain way and oriented in a certain direction. Therefore, this map already has some of the observer's perspective implicit in it. Moreover, as we have seen, even an observer with a given velocity has, at each moment, a different perspective on the universe, because he has information only about his absolute past, which corresponds to a different region of space-time in each moment of such an observer's existence. Thus, whether we consider what is seen by different observers or by the same observer at different times, it is necessary continually to relate the results of all these observations, by referring them to a space-time map with a correct structure, and in this way to develop an ever-growing knowledge and understanding of what is invariant and therefore not dependent on the special perspective of each observer.

It is seen then that while relativity does emphasize the special role of each observer in a way that is different from what is done in earlier theories, it does not thereby fall into a kind of "subjectivism" that would make physics refer only to what such an observer finds convenient or chooses to think. Rather, its emphasis is on the hitherto almost ignored *fact* that each observer does have an inherent perspective, making his point of view in some way unique. But the recognition of this unique perspective serves, as it were, to clear the ground for a more realistic approach to finding out what is actually invariant and not dependent on the perspective of the observer." (emphasis in the original)

Clearly Suntola has aimed to clarify what is actually invariant, and what varies, pursuing an ever more "realistic approach" based on his studies on the constraints and affordances of an expanding $S^3(r_0)$ geometry. The model, even if imperfect and incomplete in places, should certainly find its place in the discussions about the cosmos[47], not least due to its apparent minimalism, even elegance, in attaining these results (referring to the various metrics in Table I, for example).

VI. CONCLUSIONS ABOUT THE APPROACH

Intuitively, $S^3(r_0)$, where r_0 is the expanding radius, seems to have all the right basic constraints for a cosmological model, and not too many degrees of freedom from the start. It seems to have an almost trivial connection to the expanding universe, with singularity baked in, without having to try to infer the emergent geometry from some field theory. It does not mean that a generally covariant field theory cannot be created from this work (and I commend any kind of theoretical or experimental work studying these propositions), but it means that to be able to compare the models, we should look very carefully into the assumptions (such as the nature of the speed of light and radiation, and progression and measurement of time) baked in to each and every reported observation, unfortunately.

This is quite difficult and slow work, but seems theoretically novel, and could prove out to be very rewarding in the long run.

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Appendix A: Modeling the $S^3(r_0)$ using geometric algebras

I noted that one could start translating the equations to some suitable associative algebra, and that personally I

[47] See, e.g., Abdalla *et al.* (2022) for some open questions in cosmology.

am looking forward to see some progress here in the long term. I gathered some notes on the subject of modeling these kind of higher dimensional spaces using special kind of anticommutative algebras (geometric or Clifford algebras) here in the appendix.

Briefly, following Sobczyk (2019, p. 155), one could start by defining the associative geometric algebra

$$\begin{aligned}\mathbb{G}_4 &:= \mathbb{G}(\mathbb{R}^4) = \text{gen}_{\mathbb{R}}(\mathbf{e}_0, \mathbb{G}_3) = \text{gen}_{\mathbb{R}}(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \\ &= \text{span}_{\mathbb{R}}(1; \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3; \mathbf{e}_{01}, \mathbf{e}_{02}, \mathbf{e}_{03}, \mathbf{e}_{23}, \mathbf{e}_{13}, \mathbf{e}_{12}; \\ &\quad \mathbf{e}_{012}, \mathbf{e}_{013}, \mathbf{e}_{023}, \mathbf{e}_{123}; \mathbf{e}_{0123}), \qquad \text{(A1)}\end{aligned}$$

consisting of the scalar 1, vectors, bivectors, trivectors and the pseudoscalar $\mathbf{e}_{0123}$ of the algebra. The generating orthogonal basis vectors each square to one, $\mathbf{e}_0^2 = \mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1$, and are mutually anticommutative, i.e., $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$. The ordinary Euclidean 3-space $\mathbb{G}_3$, isomorphic to the Pauli algebra of matrices, is embedded in it, and one can always choose $\mathbf{e}_0$ to be along any ray from the origin, and the 3-space is orthogonal to it. One can also identify the Dirac algebra of $\mathbb{G}_{1,3}$ to it by choosing $\gamma_0 := \mathbf{e}_0$, $\gamma_1 := \mathbf{e}_{10}$, $\gamma_2 := \mathbf{e}_{20}$, and $\gamma_3 := \mathbf{e}_{30}$, and one can complexify the $\mathbb{G}_4$ to attain $\mathbb{G}_{4,1}$ [isomorphic to algebras $\mathbb{G}_{2,3}$ and $\mathbb{G}_{0,5}$, see the various chapters in Sobczyk (2019)].

The peculiar role of the $\mathbf{e}_0$ and the two pseudoscalars (ordinary oriented 3-volume element $\mathbf{e}_{123}$ in $\mathbb{G}_3$, and the oriented 4-volume element $\mathbf{e}_{0123}$ in $\mathbb{G}_4$) can be used to transform between different kinds of representations and their duals, as also evidenced by the so called spacetime algebra (STA).

Moreover, the even subalgebra $\mathbb{G}_3^+$ of $\mathbb{G}_3$ can be identified with the unit quaternions, as choosing $\mathbf{i} := \mathbf{e}_{23}$, $\mathbf{j} := \mathbf{e}_{13}$, and $\mathbf{k} := \mathbf{e}_{12}$, (along with the unit scalar 1), automatically satisfies the relevant quaternion multiplication rules. I first learned about these from Sobczyk (2013, 2019), but they are (or should be) widely known in the mathematics and physics communities.

A general quaternion $\alpha \in \mathbb{Q} \subset \mathbb{G}_3$ then has the form

$$\alpha = \alpha_0 + \alpha_{23}\mathbf{e}_{23} + \alpha_{13}\mathbf{e}_{13} + \alpha_{12}\mathbf{e}_{12}, \qquad \text{(A2)}$$

where α_0, α_{23}, α_{13}, $\alpha_{12} \in \mathbb{R}$. Its conjugate $\alpha^\dagger$ is defined by negating all the bivector parts but keeping the scalar part intact.

Using a general vector $x \in \mathbb{G}_4^1$

$$x = x_0\mathbf{e}_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3, \qquad \text{(A3)}$$

the $S^3(r_0)$ manifold $\mathcal{M}_{r_0}$ is then defined by

$$\begin{aligned}\mathcal{M}_{r_0} := \{x \in \mathbb{R}^4 \mid x^2 = \alpha_x\alpha_x^\dagger = x_0^2 + x_1^2 + x_2^2 + x_3^2 \\ = r_0^2\}, \qquad \text{(A4)}\end{aligned}$$

where α_x is a quaternion corresponding to a unique vector x on the three-sphere $S^3(r_0)$ in $\mathbb{R}^4$.

One can continue by defining the 3-dimensional tangent space $\mathcal{T}_x$ of $S^3(r_0)$, at any point x, by

$$\mathcal{T}_x := \{a \in \mathbb{R}^4 \mid a \cdot x = 0\}, \qquad \text{(A5)}$$

and proceed to define the differential mapping using the group product (not shown here) and vector derivative, so the group manifold of $S^3(r_0)$ can also be studied by the composition of linear mappings in its Lie algebra $su(2)$ at the identity $\mathbf{e}_0$ (as Sobczyk has also demonstrated in his recent work).

The usefulness of these concepts may be a bit complicated by the fact that in DU, the manifold $\mathcal{M}_{r_0}$ is dented radially near mass centers, breaking the spherical symmetry locally in interesting ways. However, the basic properties of associative geometric algebras seem certainly very useful in defining many geometric operations, such as higher dimensional rotations, as the exponentials are well defined, and so the elements of the algebras which function as scalars ($x \in \mathbb{R}$), unipotents ($u^2 = 1$), imaginary elements ($i^2 = -1$), nilpotents ($n^2 = 0$), and idempotents ($u_+^2 = u_+$), work together to define scaling, hyperbolic, Euclidean, parabolic, and idempotent rotations, respectively, in quite general but still graphic way.

Examples of each kind of elements are

- Scalars, $x^2 \geq 0$: ordinary scalars as pointlike quantities without dimensions, most often used as coordinate values scaling other geometric objects. Usually algebras can be built on real numbers, but sometimes it is preferable to use complex numbers (as real matrices can have imaginary eigenvalues) or hypercomplex numbers such as quaternions or octonions (which is not associative) directly,

- Unipotents, $u^2 = 1$: basis vectors such as $\mathbf{e}_0$ and $\mathbf{e}_1$, and the pseudoscalar $\mathbf{e}_{0123}$ of $\mathbb{G}_4$,

- Imaginary elements, $i^2 = -1$: bivectors such as $\mathbf{e}_{23}$ and $\mathbf{e}_{01}$, and the pseudoscalar $\mathbf{e}_{123}$ of $\mathbb{G}_3$, which happens to commute with all the elements in $\mathbb{G}_3$ so functions exactly like the imaginary unit of complex numbers (in that algebra),

- Nilpotents, $n^2 = 0$: multivector combinations of elements such as $\mathbf{e}_1 + \mathbf{e}_{12}$, which together square to zero. A sum of a nilpotent and a scalar can be used as a dual number performing automatic differentiation, as $f(x + n) = f(x) + f'(x)n$. The algebras themselves can be defined from the generating nilpotents or null vectors, such as when $\mathbf{a}_0^2 = \mathbf{b}_0^2 = 0$, where $\mathbf{b}_0\mathbf{a}_0 + \mathbf{a}_0\mathbf{b}_0 = 1$, then $\mathbf{e}_0 = \mathbf{a}_0 + \mathbf{b}_0$, $\mathbf{f}_0 = \mathbf{a}_0 - \mathbf{b}_0 \in \mathbb{G}_{1,1}$, and the process can be extended to arbitrarily high dimensions as shown in Sobczyk (2019),

- Idempotents, $u_+^2 = u_+$: combinations of a scalar and a unipotent such as $u_+ = (1 + \mathbf{e}_0)/2$ and

$u_- = (1 - \mathbf{e}_0)/2$, or $j_+ = (1 + \mathbf{e}_{0123})/2$, which have powerful properties as they work as projection operators and simplify power series representations due to their products vanishing, $u_+u_- = u_+(1-u_+) = 0$ (they are mutually annihilating). In Sobczyk (2019) idempotents are used very successfully to translate between the standard basis of algebras and their matrix representations (so called canonical null vector basis defining a spectral basis), unifying the two languages. Sobczyk shows how some properties of quantum mechanics can be visualized as stereographic projections from 4D to 3D, and further into 2D, facilitated by idempotent structures. For comparison, see also Farmelo (2005).

Note that in these kind of associative algebras, the exponentials are well defined, and can represent different kinds of rotations, depending on the element being exponentiated (here $\phi \in \mathbb{R}$ and commutes with the element in question, otherwise one would need to involve commutators):

$$e^{x\phi} = 1 + x\phi + (x\phi)^2/2! + (x\phi)^3/3! + \cdots, \tag{A6}$$

$$\begin{aligned} e^{u\phi} &= 1 + u\phi + (u\phi)^2/2! + (u\phi)^3/3! + \cdots \\ &= 1 + u\phi + \phi^2/2! + u\phi^3/3! + \cdots \\ &= \cosh\phi + u\sinh\phi, \end{aligned} \tag{A7}$$

$$\begin{aligned} e^{i\phi} &= 1 + i\phi + (i\phi)^2/2! + (i\phi)^3/3! + \cdots \\ &= 1 + i\phi - \phi^2/2! - i\phi^3/3! + \cdots \\ &= \cos\phi + i\sin\phi, \end{aligned} \tag{A8}$$

$$\begin{aligned} e^{n\phi} &= 1 + n\phi + (n\phi)^2/2! + (n\phi)^3/3! + \cdots \\ &= 1 + n\phi, \end{aligned} \tag{A9}$$

$$\begin{aligned} e^{u_+\phi} &= (u_+\phi)^0/0! + u_+\phi + (u_+\phi)^2/2! + \cdots \\ &= u_+ + u_+\phi + u_+\phi^2/2! + \cdots \\ &= u_+e^{\phi}. \end{aligned} \tag{A10}$$

The last form of $e^{u_+\phi}$ is peculiar and is perhaps not always defined as such [sometimes one encounters $1 + u_+(e^{\phi} - 1)$ instead], but as Sobczyk (2019, p. 139) remarks, the idempotents are "slippery objects which can change the identities of everything they touch. As such, they should always be treated gingerly with care." He notes that idempotents naturally arise in the study of number systems that have zero divisors, which is quite common (such as the ring of $n \times n$ matrices over a field, if $n \geq 2$).

As noted in the main text, in these kind of more advanced algebras, the square of a vector is simply its length squared, as

$$v^2 = (|v|\hat{v})^2 = |v|^2\hat{v}^2 = |v|^2, \tag{A11}$$

when the basis vectors are ordinary unipotents that square to one.

Also for orthogonal vectors, their sum squared is equal to the sum of their squares, as the cross-terms vanish. One of the greatest theorems of classical antiquity, Pythagorean theorem, illustrates this nicely in a noncommutative setting, where anticommutation $ab = -ba$ implies orthogonality:

$$\begin{aligned} (a+b)^2 &= (a+b)(a+b) \\ &= a^2 + ba + ab + b^2 \\ &= a^2 + ba - ba + b^2 \\ &= a^2 + b^2 \\ &= |a|^2 + |b|^2 = |c|^2 \\ &= c^2. \end{aligned} \tag{A12}$$

Conversely, commutativity $ab = ba$, denotes parallel vectors. For higher grade objects the meaning of commutativity and anti-commutativity alternates, i.e., for bivector $\mathbf{e}_{12}$, the parallel in-plane vector $\mathbf{e}_2$ anti-commutes ($\mathbf{e}_{12}\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_{12}$), and the orthogonal vector $\mathbf{e}_3$ commutes ($\mathbf{e}_{12}\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_1\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3\mathbf{e}_{12}$).

The most basic geometric double vector products can be decomposed as a multivector sum of symmetric and antisymmetric products,

$$\begin{aligned} ab &= (ab + ba)/2 + (ab - ba)/2 \\ &= a \cdot b + a \wedge b \\ &= b \cdot a - b \wedge a, \end{aligned} \tag{A13}$$

where the first term (inner or dot product) is evidently a scalar, and the second term (outer, exterior, or wedge product) an oriented bivector. Using the exponential function, the product has a convenient polar form

$$ab = |a||b|e^{\hat{B}\phi} = |a||b|\,(\cos\phi + \hat{B}\sin\phi), \tag{A14}$$

where ϕ is the angle between a and b, and unit bivector $\hat{B} = a \wedge b/|a \wedge b| = a \wedge b/|a||b||\sin\phi|$, for ordinary vectors a and b whose composing unit vector parts each square to one (note that operator precedence here is wedge products, ordinary products, divisions). We need to normalize the bivector to unit area, so that it squares properly as an unit imaginary element $\hat{B}^2 = -1$ in the exponential map, otherwise its area scales the phase proportionally.

For other kinds of objects, the exponential mapping may result in hyperbolic or parabolic behavior, as noted earlier, and it is quite common that in some algebras the scalar identity element 1 can work as a center as $e^0 = 1$, rotating from there so that the norm is conserved (in some specific settings). In matrix representations of the $2n$-dimensional algebras (where there are 2^{2n} elements), this usually means identity matrices of size $2^{2n} \times 2^{2n}$, as $\mathbb{G}_{p+1,q+1} \cong M_2(\mathbb{G}_{p,q})$.

Especially in $\mathbb{G}_3$, any bivector B can be represented by its dual vector (normal $\hat{n}$ to the plane), converted by multiplication with the algebra pseudoscalar $I_3 := \mathbf{e}_{123}$. The correspondence is visible in the example $I_3\mathbf{e}_3 = \mathbf{e}_{123}\mathbf{e}_3 = \mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$. So in $\mathbb{G}_3$, $ab = |a||b|e^{I_3\hat{n}\phi}$, which illustrates the geometric meaning of complex exponents in some physical formulations. One must remember that the pseudoscalar does not generally commute with all the other elements in other algebras than the $\mathbb{G}_3$, so usually it is best denoted by some other symbol than i, which associates so strongly with the imaginary unit of complex numbers.

The basic properties of exponential maps, such as $e^x e^y = e^{x+y}$ (if x and y commute), $1/e^x = e^{-x}$, and $2^n = e^{n \ln 2}$, complement nicely the essential feature of geometric algebras: having a multiplicative inverse for many kinds of products, even derivatives (provided the element in question is invertible, i.e., not zero norm such as a nilpotent, or a zero divisor such as an idempotent). For example, the inverse of a vector is simply pointing to the same direction but its length inverted, $v^{-1} = v/v^2 = \hat{v}/|v|$. The inverse of the aforementioned vector product $ab = |a||b|e^{\hat{B}\phi}$ is simply $e^{-\hat{B}\phi}/|a||b|$, in any dimension.

Using the alternating symmetric and anti-symmetric property of higher order products, one can derive a useful formula for rotating the projection of a vector c in the plane of a bivector $a \wedge b$ by $\pi/2$:

$$(a \wedge b) \cdot c = a(b \cdot c) - b(a \cdot c) = -c \cdot (a \wedge b), \qquad \text{(A15)}$$

which is, while true, unfortunately a bit tedious to prove. However, with it, and noting that $(a \wedge b) \wedge c = a \wedge b \wedge c$, one can derive the very illustrative simplified equation for a geometric triple vector product:

$$abc = (a \cdot b)c - (a \cdot c)b + (b \cdot c)a + a \wedge b \wedge c, \qquad \text{(A16)}$$

which is a multivector composed of a vector and a trivector part. It is clearly trilinear (as it should, the product is associative, distributive, etc., just not commutative in general), and the middle position in the product is special, with the minus sign of b appearing in the vector decomposition. If one sets $b = a$, one gets simply $a^2 c = (a \cdot a)c = |a|^2 c$, as the wedge product of a vector with itself vanishes. However, if one sets $c = a$, then the form starts to resemble the sandwich product of spinors:

$$\begin{aligned} aba &= 2(a \cdot b)a - (a \cdot a)b \\ &= |a|^2 \left[2(\hat{a} \cdot b)\hat{a} - b\right] \end{aligned} \qquad \text{(A17)}$$

and this is not a coincidence, as it describes a reflection of b across $\hat{a}$ (scaled by $|a|^2$). The form aba^{-1} is then a reflection without a scaling effect. By geometric construction, form $-aba$ means simply the reflection of b across the hyperplane defined by a as a normal to the plane (in any dimension). It is very interesting that sandwiching any multivector between a vector, nMn, always preserves the grade of the multivector M, as the other one brings back that which the other one has potentially collapsed, and vice versa, if the other one has brought in a new direction, the other one folds it away.

Rotations (with possible scaling) can be depicted by nesting two reflections, such as $dabad$. One can use the exponential form, inverses and square roots to arrive at a generic rotation $R\,b\,R^{-1} = e^{\hat{B}\phi/2}\,b\,e^{-\hat{B}\phi/2}$, where R is the rotor $e^{\hat{B}\phi/2}$ and $\hat{B}$ is a unit bivector in the plane of $d \wedge a$, the combination representing rotation of b by angle ϕ in the plane, in any dimension. The odd and even parts are handled together, and the commuting and anti-commuting components of b, with respect to its rejection and projection in the plane, work together to compose the rotation.

When the exponential mappings involve other kinds of geometric objects than unit bivectors (or when we are operating in an algebra where unit bivectors do not necessarily square to -1), the "rotations" can be quite complicated (such as hyperbolic boosts, or more complicated combinations), but they may still have physical uses and relevance.

Lastly, one can use a similar identity than Eq. (A15) to project a vector d into the bivector components of a trivector [see Eqs. 4.49 and 4.50 in (Doran and Lasenby, 2003, p. 94)],

$$(a\wedge b\wedge c)\cdot d = (a\wedge b)(c\cdot d) - (a\wedge c)(b\cdot d) + (b\wedge c)(a\cdot d). \qquad \text{(A18)}$$

Using it, one can decompose a formula for a geometric product of four vectors $abcd$, as

$$\begin{aligned} abcd \;=\; & (a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (b \cdot c)(a \cdot d) \\ & + (a \cdot b)(c \wedge d) - (a \cdot c)(b \wedge d) + (b \cdot c)(a \wedge d) \\ & + (a \wedge b)(c \cdot d) - (a \wedge c)(b \cdot d) + (b \wedge c)(a \cdot d) \\ & + a \wedge b \wedge c \wedge d. \end{aligned} \qquad \text{(A19)}$$

First line is a scalar, last one is a quadrivector, and the middle lines are the bivector parts. Note that in $\mathbb{G}_4$ and higher dimensions, not all bivectors have simple expressions—two planes do not necessarily meet in a line, such as $\mathbf{e}_{01} + \mathbf{e}_{23}$, so in that case the sum cannot be written as the exterior product of two vectors. Of course, simply by looking at the various symmetries present in the above, there are certainly many kinds of forms that the product could be written in, highlighting its different features.

Following the rather profound ideas exemplified in the geometric products of ab and abc, I would like to understand more about the $abcd$ product, too (especially in $\mathbb{G}_4$ or higher, where the quadrivector seldom vanishes).

Only recently I got to know about Doran and Lasenby (2003), which could be of great value in modeling the expanding $S^3(r_0)$ geometry, especially its chapters on geometric and multivector calculus. Also its other parts could prove out to be enlightening for these studies, as the connection to physics is so direct there, and the book seems to also combine spacetime algebra with the gauge treatment of gravity.

I will end this appendix with a praise from J. C. Maxwell towards W. K. Clifford, in the year 1870, quoted in the same volume as (Lounesto, 1986), p. xiii:

> "The pecularity of Mr. Clifford's researches, which in my opinion points him out as the right man for a chair in mathematical science, is that they tend not to the elaboration of abtruse theorems by ingenious calculations, but to the elucidation of scientific ideas by the concentration upon them of clear and steady thought. The pupils of such a teacher not only obtain clearer views of the subjects taught, but are encouraged to cultivate in themselves that power of thought which is so liable to be neglected amidst the appliances of education".

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